Polytope Names and Constructions

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Abstract

A new multi-dimension version of the Kepler-style names for the Uniform-edge and Uniform-Margin polytopes.

1 Introduction

There is little comfort in complaining about the lack of a clear terminology for the higher dimensions. But instead of doing this, I intend to create a set of terms that span the dimensions comfortably. The fault here lies in that common words have different meanings that belong to objects of different dimensions outside of three dimensions.

A line in the sand, a dead-line, the front lines, to toe the line, are divisions of space. In a land of four dimensions, the surface of a planet is three dimensions, and in four dimensions increase a dimension, to keep pace with solid space. The bee line, the railway line, the bus-line, are trips from point to point, and do not increase dimension.

The common pattern is to suppose that the dimensionality of the 3d case is correct, and invent new terms for relative to solid. To this end, we get a *facet* having many *faces*, since the facet has moved up a dimension, while a face has not. A projection of the Schägel-diagram of a polychoron (4d polytope), presents itself as a foam of the surface elements, a foam of cells, so to speak. Cell is elsewhere used to represent a room in a foam or tiling, and it is no good to extend the meaning to include include specific elements of a polytope.

A plane is a dividing space. Mathematically, we might represent a plane as one equal-sign, viz $z = 0$. In 3d, this is where our descent under gravity ends, and in higher dimensions, the descent against gravity is best represented by $z = 0$, or one equal-sign, regardless of how many dimensions there are. One equal-sign divides space.

The armies that surround cities do so, by forming a solid shell in the plane. They do not form any cover over or under the city, but follow the city limits. It's a matter of two equal signs ($z = 0, r = 0$), which divides the surface of the planet into an 'inside' and 'outside'. The terms inside and outside have meaning only in terms of the object is *solid.* Thus the surface represents the bounding limit of a solid.

The dancers do so *around* the maypole. The maypole is vertical, but the dancers do not invade its space (which is the vertical line that contains the pole). Instead, the action happens in a space that is orthogonal to it: the ground. We use the terms like 'around' and 'aroundings' around such spaces.

Stems deriving from *face* are held to denote fragments of spaces of one equal-sign. So when one is facing off against another, the intent is to block all routes, like a wall.

Although one might suppose a line is made of points, and a 2-space (hedrix) of lines, and so forth, the reality is that these are derived from the intersection of planes. In three dimensions, a point is the crossing of three planes, and so has three equal signs. The spaces of fixed dimensions have new names, we we give in the next section.

2 The Fabric of Space

The word *polyhedron* is reanalysed as three stems, *poly·hedr·on.* Since *hedron* refers to the face of a polyhedron, the word is read as if to mean a *closed bag · made of 2d · patches.*

Supposing this, we invent the suffix *ix* to denote a fabric that the patches might be cut. So *hedra* “2d oatches” are cut from a *hedrix* “2d cloth”. The nature of the cloth is that it is nominally unbounded. That is, we are not to find any limits to the cloth for the applied end. It can also refer to the a full unbounded (aperific) extent.

By replacing various parts of the stem, we derive a more extensive range of names for the higher dimensions. Using the stem *chor* for *hedr*, the expression becomes *3d* fabric and patches. A polychoron is a solid in 4d, specifically a closure of 3d patches. A set of names is provided for dimensions 0 to 8.

Teel A fabric of zero dimensions, such as a button. Teel is related to the greek *telos* “journey, destination”. Since “tele-” is already an active stem, the vowel is lengthened, to denote the destination. A *teelic infinity* is a model which supposes the destinations of numbers is less than the path, such that 1+3 = 2+2 both end at 4.

Latr A fabric of one dimension, such as a thread.

Hedr A fabric of two dimensions, such as a cloth. The word hedr relates to a seat, the illusion that a dodecahedron might make a beanbag. Cat·hedral is the over-seat of the church.

Chor A fabric of three dimensions, such as a brick. It is related to *camera*, *chamber*. The space we live in is a *horochorix* ‘horizon-centred 3d fabric’.

Tera, Peta, Ecta, Zetta, Yotta The fabrics of 4, 5, 6, 7, and 8 dimensions. They are the metric prefixes representing $1{\cdot}10^{3n}$, the fabric from a line of a kilo-dot, would have a tera-dot, peta-dot, etc points. The correct prefix for 6d would be *exa*, the resulting fabric is exix. But since this would dissolve to *ectix*, the stem ect- was regularised throughout.

Replacing *poly* with other stems, provides us with words to mean an assembly of patches, not necessarily closed, such as a *multi*hedron (such as the net of a cube).

Apeiro- and *peri* ate derived from the greek, eg apeiron "boundless, as a sea or desert". A perimeter or periphery is a limit that contains the object of interest. It happens in the (sub-)space where the object is *solid*. Where the object might be contained within a patch of the space, it is bounded. A tiling is evidently unbounded, and so is an *apeirotope*, but in some spaces, even all-space is bounded.

Infinito is used to represent without number. A winding of a long chain around a spool makes for the prototype of an infinitolatron.

3 The Products

To be a product, there ought be a mathematical mapping of some property, that the property of the product is the product of the properties (of the factors). Each of the five regular solids in every dimension defines a product.

The **surtope** products use the surtope-count as the product-property, the resulting product is of the same form as the factors.

Repetition Products of repetition make a copy of the factor at each point of the co-factor. The cube is an example, for at each point of height, the section is a copy of the base square. Likewise, one might imagine for each point of the square base, there is a copy of the height.

Draught The products of draught is made by drawing a line AB between the points A of one base, and B of the second. the original elements are kept. An addition to the surtope equation of an element 1 is made to the right, that point × point = line. A product of draught increases the dimension.

Content In the product of content, the whole of the element’s surface and interior are used in the product. For this to work, an element 1 is added to the left of the surtope equation, to stand for the interior.

Surface The product of surface is such that the content of the factors are not counted in the product, instead, the surface of the product is the product of the surfaces. A product of surface reduces the dimension. The draught of surface increases and decreases the dimension by 1, leaving the dimension the sum of the factors’.

The **coherent** products use the content-measure as the product-property, the content of the product is the product of the contents. It is called 'coherent', because the product-powers of a unit line defines the units of higher content. The square and cubic measures are examples of this.

Radiant The radiant products suppose that the surface of the solid represents a value of 1 in every direction, and that for all other points, it is a multiple of the distance from the centre 0 to the surface 1. A radiant of $\frac{1}{2}$ represents a surface of a copy $\frac{1}{2}$ of the size.

The products of elements X, Y, Z, are represented in cartesian coordinates as x, y, z, the surface being as some function of these. For example, the prism product is represented as $\max(x, y, z)$. Note that this value still produces a radial value, and the surface of the product is also when it is equal to 1.

3.1 PRISM = repetition of content = max()

Prism is derived from the Greek word for *offcut*. Such might be imagined that one has a hexagonal bar, and from it cuts equal measures of length. The result is hexagonal *offcuts* or prisms. In general, one might suppose that where the points are marked as belonging to a factor of the product, the prism is the intersection of the various spaces for the marked areas.

The canonical cube is the product of the line-segment (-1, 1), which leads to the coordinates $\pm 1, \pm 1, \ldots$. The radiant function is represented by abs x_i, the surface is formed when any one of these equals 1.

The radiant product here is $\max(b_1, b_2, \cdots) < 1$. It provides coherent units represented by the measure-polytope (square, cube, tesseract, ...) of unit edge.

The surtope adds an element to the left only, so a cube = 6h 12e 8v becomes 1c 6h 12e 8v being (1e 2v)^3. This equation might be written without the identifiers c = choron (3d) h=hedron (2d), e=edge (1d), v=vertices (0d), as 1.2.# ^{3}= 1.6.12.8.#. The hash # tells us that this item is not used in the calculation.

3.2 TEGUM = draught of surface = sum()

Tegum is derived from a Latin word for *cover*. It is related to *toga*, and *thatch*. The tegum provides by draught, a cover for the new interior, by drawing[1] points of surface from each element.

The canonical tegum is the rhombus, octahedron, 16choron, etc. This is the tegum-product of the lines (-1, 1) on each axis, the radiant function is again abs 1, the surface given by $\text{sum}(x_1, x_2, \ldots) = 1$.

The surtope consist is augmented by no content term #, and a term to the right for the nulloid[2].

The octahedron has 8 hedra, 12 edges, and 6 vertices, or 8,12,6. The tegum-form is to enclose this in #, 1, as #,8,12,6,1. This is the cube of #,2,1, which is a line in tegum-form.

There are no general-use units for this as yet. The regular cross-polytope is the tegum-power of its diagonals, and thus for a cross-polytope of unit edge, for having a diagonal of $\sqrt{2}$, has a volume of $\sqrt{2}^n$ in tegum units.

However, the series of units is coherent with the definition of content as the moment of surface, that is, $C = \int \mathbf{r} \cdot \mathrm{d}\mathbf{S}$. Taking the origin to be the corner of a cube, the content of a cube is n times its face, and by recursion the measure-polytope is $n!$ times the tegum-product.

3.3 CRIND = rss()

The circle, sphere, glome, represent a class of regular solid (although not a polytope, it does have a hard surface), as such might be represented by the product of its diameters. Varying the diameters give rise to a family of ellipses and ellipsoids.

The canonical sphere is $x_1^2 + x_2^2 + \ldots = 1$, represented again by the diameters [-1,1] in each axis. Putting these axies to different values gives rise to ellipsoids.

It ought be recalled that ordinary folk measure circles by the diameter, and not the radius. As such, an eight-inch plate has a diameter of eight inches. A *circular inch* is the area of a circle, the diameter of

[1] Draw as in to draw glass or what chewing gum does when separated

[2] The nulloid is the lower point of incidence, representing a dimension of -1. In draught-products, the dimension-number is increased by 1 to match the vertices of the simplex.

which is one inch. Such were used before calculators, to eliminate π from calculations, when it was not really needed.

For measuring volumes, the typical unit is a *cylinder inch*, being a cylinder of unit height and base. The proper coherent unit is a *spherical inch*, being a sphere of unit diameter, 2 cylinder inches = 3 spherical inches.

3.4 PYRAMID = draught of content

The simplexes are the pyramid-power of its vertices.

The canonical simplex is represented by the points (1,0,0,0..), (0,1,0,0..), representing a face of a cross-polytope of higher space. The plane is represented by an $n+1$ space, of points of a common sum (here 1). By using a different sum for the coordinates, it is possible to shift the points around, and still keep the same lattice.

The product adds a dimension for each time the product is applied. So the product of two lines gives a tetrahedron, the rectangular sections give x% of one base times y% of the other base, the variance in x, y are not in the lines, but in the height or *altitude* of the figure.

The volume of the regular simplex is derived from the moment of the face. The point closest to the centre is $\frac{1}{v}$ on the plane, and the length of this in every axis, $(\frac{1}{v}, \frac{1}{v}, ...,$ gives $\sqrt{1/v}$. The volume of the part in the all-positive section is 1, in tegum measure, and thus the volume of a simplex in v vertices, of edge $\sqrt{2}$, is $\sqrt{n}$. From this we find the volume in prism-units to be $\sqrt{n+1}/\sqrt{2}^n n!$

The Pyramid surtope form adds a '1' at each end, so a line is 1,2,1, being a point (1,1) squared.

3.5 COMB = repetition of surface

The comb product is a product of at least polygons, including the euclidean line-tiling (horogon[3]). in the case of polygons it forms a tunnel or *comb*, in the sense of tilings, such are also called honey*combs.*

The canonical tiling is the euclidean grid of integers, represented by the powers of the number-line. The corresponding powers of the number-line gives rise to the square, cubic, tesseractic, tilings. One can use other tilings in this process: the hexagonal - horogon tiling is a tiling of hexagonal tiles.

In hyperbolic space, this product still exists, but the horogon is the primitive or first power. The powers are still bounded by squares, cubes, etc, four at a margin, but it no longer exists in a cartesian coordinate system.

The second form is to produce toruses. The regular torus itself is the comb-product of two circles, the larger circle, and a smaller circle representing the cross-section. This might be polytopised by replacing the circles with polygons, such that one has a bent column, made of little pyramid-sections. Note there is no rotation in the comb-product.

In four dimensions, it is possible to have a decagon-dodecahedral comb. A hollow tower is made of pentagonal prisms, the base fitted together to form a dodecahedron, the height being ten units high. It can be converted into a torus in two different ways.

sock In this method, one supposes that a bar (like the leg), runs down the centre of the tower. The tower is then peeled outwards as one takes off a sock, rolling down until it connects with the base.

hose This method connects the top to the bottom by bending the bar into a circle, such that the two join, as one might connect the ends of a hosepipe.

The products produce distinct items. The first is the result as if you poked a line through a glome, giving the equal of a hollow-sphere slice. A string passed through this hole will form a link that one might lift it.

3.6 Bracket-topes and Coherence

The three coherent products are represented by the brackets [Prism], (Crind) and <Tegum>. These are applied over a set of perpendicular lines, represented by letters, using 'i' as the default. The brackets

[3] The Horogon is a horizon or infinite-centred polygon, the edges are orthogonal to rays that converge on the horizon. Other infinite polygons exist in hyperbolic space, such as the bollogon, whose edges are perpendicular to orthogonals of a straight line

might be nested, but a parent can absorb a direct child bracket if they match, so ((II)[II]) = (II[II]) = circle-square crind.

In three dimensions, one might, apart from the regulars [III] cube, (III) sphere, <III> octahedron, have a variety of other bracket-topes, such as [I(II)] cylinder, (I[II]) square crind, and <I(II)> bi-cone. The square crind is the intersection of cylinders at right-angles to their height.

The products are coherent to their own set of units, and thus it is possible to find the volume of a bracket-tope by way of unit-changing. For example, the volume of a square crind (I[II]) is first to find [II] = 1 P2, and convert this into C2 units. π P2 = 4 C2, so the area of [II] is $\frac{4}{\pi}$ C2. Multiply this by C1, and we get $\frac{4}{\pi}$ C3. Since C3 = $\frac{\pi}{6}$ P3, the volume is $\frac{pi}{6}\frac{4}{\pi} = \frac{2}{3}$ P3 units.

Note that it is not correct to put these units in the same product. This is because arithmetic multiplication maps onto three entirely different products. The product covering P2C1, for example, does not state the overall parent, which could be P or C (or even T). However, it is correct to put P2P2 = P4 as a matter of coherence.

The ratio of volumes run Pn/Tn = $n!$, Pn/Cn = $n!!/(1,\pi/2)^n$, and Cn/Tn = $(n-1)!!(1,\pi/2)^n$. The factor $(a,b)^n$ corresponds to an alternating power, that is, the first n items in the list $a, b, a, b, a, b....$ The double-factorial is a descent from the value, such that the value is always greater than zero. So $7!! = 7 \cdot 5 \cdot 3 \cdot 1$.

P/C runs $(1) = 1,\ (2) = 4/\pi,\ (3) = 6/\pi,\ (4) = 32/\pi^2,\ (5) = 60/pi^2,\ (6) = 384\pi^3,\ (7) = 840/\pi^3$

C/T runs $(1) = 1,\ (2) = \pi/2,\ (3) = \pi,\ (4) = 3\pi^2/4,\ (5) = 2\pi^2, (6) = 15\pi^3/8, (7) = 6\pi^3$.

4 Kepler-style constructions

Progressions are transformations from one polytope to another. It can be as simple as scaling, as we have met in the radiant products. New faces might be formed as the older faces separate. Such might be various prisms or pyramids (that is, the content products), or a pyramid erected on a slice (such as converting a line to a square, giving a triangular prism. Other progressions might represent the time scale of some dynamic process, or a convex hull thrown over a compound of like figures.

4.1 Antiprisms

The largest class of uniform figure, not derived from regulars or their prisms, is the antiprism. These exist for all polygons, and consist of two identical polygons, one rotated by half an edge. In between is a row of triangles, and a set of edges zig-zaging from top to bottom and back.

Such zigzag is reminiscent of the lacing on a drum, or a shoe, which does exactly this between the top and bottom, or the two sides that close on a shoe. Since many lace prisms are made by defining parallel sections, and lacing these together, it is a suitable term for such compound-connections.

The general antiprism is taken as two polytopes in dual position. For each surtope of the top, there is a matching surtope of the dual at the bottom, these in the regular instance would be fully perpendicular at the centre of each. In the antiprism, these are set in pyramid product, the progression of height converts these into prisms of the matching surtopes, one increasing and one decreasing until exhaustion.

The **antiprism sequence** is the expansion of a polytope, such that the original faces are kept. There forms prisms between each face, a margin-line prism, and so forth until the vertex, which is replaced by the faces of the dual. Because these elements are orthogonal, these are not restricted to any shared symmetry: in the 24chora, triangle-line prisms form between the faces, and line-triangle prisms along the former edges. The vertices become the dual of the vertex-figure, or the face of the dual, giving octahedra.

This sequence is usually one of the first to be seen.

The tegum product of antiprisms, is itself an antiprism. If Aa, Bb, ... represent the axies of the antiprism, the upper and lower cases are duals, then there is a pyramid face ABC... opposite a pyramid face abc.. as an antiprism. It follows also that any case pattern can be used, eg Abc.. vs aBC.. The same polytope can be antiprisms to many different figures.

4.2 Antitegums

The dual of an antiprism, is an *antitegum*. It exists as a regular construction from polygons for all numbers. Such is formed by the intersection of *lace cones*, in this case, the cones are point-pyramids of the duals, the expanding portion of one intersects with the contracting portion of the other. One might suppose two people are shining lights at each other, the light projecting a perfect pyramid of the filter at the light. Where two triangles are used, and rotated opposite each other, a cube would arise.

Lace Cones can be best seen in polytopes such as the tetrahedron and cube. In the case of the cube, imagine that the three faces around a vertex are *red*, and those around the opposite *blue*. The red faces would extend to a full octant of space, as would the blue. But for the intersection, we see that the red light ends that of the blue and vice versa. In the case of the tetrahedron, we see that one could imagine two red faces meeting two blue. The section here is a simple 'V' shape. However, this is not solid, and so is extended in all directions perpendicular to the V.

Likewise, three red faces and a blue face, is the intersection of light-cones from a triangle and a point. The triangle is solid in 3d, but to render the point, we need to expand it in all directions perpendicular to the antitegum axis. The dual of pyramid products of all kinds, are by the intersection of solid lace cones of the dual of the bases.

The **antitegmic sequence** is the expansion of one figure, intersecting the reduction of the dual. The sequence forms the families of *truncates* and *rectates*, the truncates are as the intersection is consuming the n-surtopes (vertex, edge, &c), while the rectates are when this surtope has been fully consumed, and the vertex is standing at the centre of it.

The **Hasse Antitegum** is the incidence diagram of the base. Against the axies, the hasse antitegum provides layers of vertices, one for each surtope. A surtope is incident on another if the representing vertices fall on the same surtope of the antitegum. All of the surtopes of an antitegum are antitegums, and so an incidence represents the long axis of some lesser antitegum.

When the diagonal is taken to the bottom of the full antitegum, the incidence is between the surtope and nulloid[4]. The top-most vertex represents the content. Between these are the added '1's that we make in the various products. It is also the source of the additional '2' in Euler's characteristic equation for odd dimensions. For example, the cube gives $6 - 12 + 8 = 2$, for having left out two terms of -1, one at each end.

The hedra of the antitegums are always rhombuses. If some surtope $n + 1$ is incident on some $n - 1$, there are exactly two surtopes n incident on both. This is what Norman Johnson means by a *dyadic* polytope, since the rhombus by itself is the Hasse antitegum of a line-segment or *dyad*.

4.3 Truncation and Rectification

The truncation and rectification is provided by the intersection of the descent of the dual. We suppose the outer is descending on the inner, both retaining their common centre and symmetry.

When the surfaces first meet, the vertices of the inner just touch the faces of the outer. This is the *zero-rectate*, the proceeding where the inner expands to meet the outer, is the *zero-truncate*. As the vertices emerge, they are cut off or *truncated*. The new vertices seek to shorten the old edges, and a new face is formed at the old vertex. This continues to the first *rectate*, where the outer's edges have been shortened to zero and the vertices meet in pairs.

As the outer continues to descend, the vertices head towards the centres of the polygon-elements. This is the *second truncate*, ending when the vertices join up in the centre of the 2d element (at the *rectate*). This continues until the n truncate, where the outer polytope has passed through the surface, and and all is left is the outer-polytope shrinking to vanish at the centre (n-truncate).

The antitegum-sequence is the time sequence of the truncates and rectates. It can be seen that there are a pair of lace-cones which represent a point-inner pyramid expanding to the left, and a second point-outer contracting to the right.

The duals of these is a similar process, except that we imagine that a rubber sheet covers the polytope, and the resulting figure is the hull of the inner and outer parts.

As the inner part expands from zero, it is the zero-apiculate, ending in the zero-surtegmate. As the inner figure crosses the surface of the outer one, the old faces of the outer figures are replaced by pyramids, whose apices are the vertix of the inner one and the margins (wall between faces) of the outer. This is the first apiculate.

[4]The Nulloid is taken as a surtope of -1 dimensions. It is incorrectly associated with the empty set, for being part of every surtope. But it's not a part of surtopes that are not parts of the polytope, and its existence is a mark that these various elements have been brought into a unity

The first surtegmate happens when the pyramids line up in pairs, and we have a tegum-product of the edges (E1)[5] of the inner one and the margins (M1). Where first the faces were pyramids against the vertex, they now come to be pyramids against the edges of the inner figure, and M2 of the outer.

The second surtegmate comes when the polygons of the inner figure have broken to surface, while the M2 of the outer ones are visible, so The faces are tegum-products of E2 of the inner and M2 of the outer, and so forth.

4.4 Cantelates and Cantetruncates

The first-truncations and first-rectification of a n-truncate gives the n-cantetruncate and n-cantelates. The duals have no special construction or name. The term is borrowed from Norman Johnson.

4.5 Runcinates and Strombiates

The process of runcination is to push the faces outwards, without changing the size of the faces. As the faces separate, the convex hull creates new line-prisms on M1, E2-M2 faces, all the way to the vertex. This becomes the face of the dual. Allowing the original faces to shrink to nothing, causes the runcinate to turn into the dual of the figure.

The dual figure is the strombiates. Imagine you have an polytope, and then draw on its surface, the elements of its dual, as would be projected by an central lamp. The faces are divided into something that preserves the face-vertex line, and all flags there-attached. You can push one in relative to the other. The name comes from the faces of the figure are antitegums of the vertex-figure of the faces of either, which are duals at each end of the vertex-face line.

The sequence of runcinations leads to the antiprism of either of the duals.

The bulk of faces of a runcinate are prisms of a surtope and its matching arounding of the dual. This gives a cycle of prisms, which leads to my old name for it (prism-circuit), and Jonathan Bower's -prismato- infix. The simplex prism circuit, or runcinated simplex, is the vertex-figure of the tiling `An`.

4.6 Omnitruncate and Vaniate

The simplex represented by the centres of each surtope, is taken as a simplex $v_0, v_1, v_2, ...$, is called a *flag*. If the rays from the centre are adjusted so that these flags do not align with any neighbouring flag, then this is the *vaniated* polytope, meaning, its flags are made into faces.

The omnitruncate corresponds to having a vertex in the interior of the flag, in such a way that edges need to be dropped to its images in any adjacent flag. This result gives the Cayley diagram for the group, that is, each kind of operation on the group is met by a walk from vertex to vertex of the omnitruncate.

5 Developments

A development here represents a change of the structure of a solid, to allow its representation. Such are the art of the modeller. In such, these represent various adjustments to model something that is not directly rendered as a model.

Atom A packing of spheres to resemble a chemical lattice. The models of atoms showing bonds are more a case of a spheration of the situation.

Bevel To act as to plane away sharp edges, to leave more rounded elements for a surtope. An example is an edge-bevelled cube, where the vertices and edges are replaced by elongated hexagons.

Frame The surtopes up to a given level, such as edges. The most common form is to provide a see-through presentation of a polytope. A hedral frame of four dimensional polytopes, as projected onto three dimensions, looks like a foam of cells, whence the misuse of the word 'cell' for face.

[5] The style here is to count surface polytope as edges of given dimensions, eg E0 for vertex, E1 for edge, E2 for hedra, and so forth. Likewise, the down-count is to count M0 for the face, M1 for the margin, M2 for the second-margins (ie n-3 element. For a polytope of n dimensions, the Mm is E(n-m-1). In 3d, a polyhedron has M0 = E2 = polygon, M1 = E1 = line, M2 = E0 = point.

Periform The stem 'peri' is allocated to mean the outmost limit. The five-pointed mullet[6] is mathematically a zigzag decagon, is the periform of the pentagram. Even so, the stitching of these mullets onto flags might include the proper edges of the polygram.

Spheration This is to replace vertices and edges with spheres and pipes, as much as if a sphere had been run along every point of these items. ZomeTools produce a spherated edge-frame of polytopes.

Surtope Paint A notional paint or glitter, sprayed onto a curved fabric, would produce a map of surtopes of the same topology. Applying more paint makes the surtopes smaller. For example, a cone gives rise to a pyramid, the more paint increases the number of edges at the base.

6 Progressions

A progression is an alteration of a polytope or solid, by means of increasing or reducing the surtope by a solid product (prism or pyramid), such that it might change to a second polytope. Such a progression is usually in a line from A to B where these are taken to be separate layers.

The idea behind progressions might be seen with the sectional layers of polytopes. A point expands to an icosahedron, and this becomes an apiculated dodecahedron, and so forth. It is noted that the convex hull overall may be larger at a given layer, than the arrangement of vertices suggest. This is because uncompleted surtopes are still running.

That one polytope can progress to another is demonstrated by the simple expansion from a point.

6.1 Progression-space

For each axis of some space, each point represents a state of some figure in progression. The simplest case might be size, but operations like runcination (a series of increasing size and surtope bevelling such that the original surtopes are unchanged), are equally valid processes.

An additional axis is provided, representing the altitude, or point in an orthogonal space where the action might be said to happen. From this a progression-polytope might be constructed by taking at each point of the altitude, a prism-product of the progressed elements.

Altitude	**Axis 1**	**Axis 2**	**Axis 3**
$(1, 1, 0)$	triangle	line	point
$(1, 0, 1)$	triangle	point	line
$(0, 1, 1)$	point	line	line

Such represented the earliest implementation of what would become a lace structure. Because at each point of the altitude, it is a prism-product, the appearance of a point represents the identity element. Without this point, the product would be zero. With the point, it appears as having no section in that axis.

[6] A mullet in heraldry is the 'stars' one sees on flags and the like