Polytope Names and Constructions

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Abstract

A new multi-dimension version of þe Kepler-style names for þe Uniform-edge and Uniform-Margin polytopes.

1 Introduction

Þere is little comfort in complaining about þe lack of a clear terminology for þe higher dimensions. But instead of doing þis, I intend to create a set of terms þat span þe dimensions comfortably. Þe fault here lies in þat common words have different meanings þat belong to objects of different dimensions outside of þree dimensions.

A line in þe sand, a dead-line, þe front lines, to toe þe line, are divisions of space. In a land of four dimensions, þe surface of a planet is þree dimensions, and in four dimensions increase a dimension, to keep pace wiþ solid space. Þe bee line, þe railway line, þe bus-line, are trips from point to point, and do not increase dimension.

Þe common pattern is to suppose þat þe dimensionality of þe 3d case is correct, and invent new terms for relative to solid. To þis end, we get a *facet* having many *faces*, since þe facet has moved up a dimension, while a face has not. A projection of þe Schägel-diagram of a polychoron (4d polytope), presents itself as a foam of þe surface elements, a foam of cells, so to speak. Cell is elsewhere used to represent a room in a foam or tiling, and it is no good to extend þe meaning to include include specific elements of a polytope.

A plane is a dividing space. Maþematically, we might represent a plane as one equal-sign, viz $z = 0$. In 3d, þis is where our descent under gravity ends, and in higher dimensions, þe descent against gravity is best represented by $z = 0$, or one equal-sign, regardless of how many dimensions there are. One equal-sign divides space.

Þe armies þat surround cities do so, by forming a solid shell in the plane. Þey do not form any cover over or under þe city, but follow þe city limits. It's a matter of two equal signs ($z = 0, r = 0$), which divides þe surface of þe planet into an 'inside' and 'outside'. Þe terms inside and outside have meaning only in terms of þe object is *solid.* Þus þe surface represents þe bounding limit of a solid.

Þe dancers do so *around* þe maypole. Þe maypole is vertical, but þe dancers do not invade its space (which is þe vertical line þat contains þe pole). Instead, þe action happens in a space þat is orþogonal to it: þe ground. We use þe terms like 'around' and 'aroundings' around such spaces.

Stems deriving from *face* are held to denote fragments of spaces of one equal-sign. So when one is facing off against anoþer, þe intent is to block all routes, like a wall.

Alþough one might suppose a line is made of points, and a 2-space (hedrix) of lines, and so forþ, þe reality is þat þese are derived from þe intersection of planes. In þree dimensions, a point is þe crossing of þree planes, and so has þree equal signs. Þe spaces of fixed dimensions have new names, we we give in þe next section.

2 Þe Fabric of Space

Þe word *polyhedron* is reanalysed as þree stems, *poly·hedr·on.* Since *hedron* refers to þe face of a polyhedron, þe word is read as if to mean a *closed bag · made of 2d · patches.*

Supposing þis, we invent þe suffix *ix* to denote a fabric þat þe patches might be cut. So *hedra* "2d oatches" are cut from a *hedrix* "2d cloþ". Þe nature of þe cloþ is þat it is nominally unbounded. Þat is, we are not to find any limits to þe cloþ for þe applied end. It can also refer to þe a full unbounded (aperific) extent.

By replacing various parts of þe stem, we derive a more extensive range of names for þe higher dimensions. Using þe stem *chor* for *hedr*, þe expression becomes *3d* fabric and patches. A polychoron is a solid in 4d, specifically a closure of 3d patches. A set of names is provided for dimensions 0 to 8.

Teel A fabric of zero dimensions, such as a button. Teel is related to þe greek *telos* "journey, destination". Since "tele-" is already an active stem, þe vowel is lengþened, to denote þe destination. A *teelic infinity* is a model which supposes þe destinations of numbers is less þan þe paþ, such þat $1+3 = 2+2$ boþ end at 4.

Latr A fabric of one dimension, such as a þread.

Hedr A fabric of two dimensions, such as a cloþ. Þe word hedr relates to a seat, þe illusion þat a dodecahedron might make a beanbag. Cat·hedral is þe over-seat of þe church.

Chor A fabric of þree dimensions, such as a brick. It is related to *camera*, *chamber*. Þe space we live in is a *horochorix* 'horizon-centred 3d fabric'.

Tera, Peta, Ecta, Zetta, Yotta Þe fabrics of 4, 5, 6, 7, and 8 dimensions. Þey are þe metric prefixes representing $1{\cdot}10^{3n}$, þe fabric from a line of a kilo-dot, would have a tera-dot, peta-dot, etc points. Þe correct prefix for 6d would be *exa*, þe resulting fabric is exix. But since þis would dissolve to *ectix*, þe stem ect- was regularised þroughout.

Replacing *poly* wiþ oþer stems, provides us wiþ words to mean an assembly of patches, not necessarily closed, such as a *multi*hedron (such as þe net of a cube).

Apeiro- and *peri* ate derived from þe greek, eg apeiron "boundless, as a sea or desert". A perimeter or periphery is a limit þat contains þe object of interest. It happens in þe (sub-)space where þe object is *solid.* Where þe object might be contained wiþin a patch of þe space, it is bounded. A tiling is evidently unbounded, and so is an *apeirotope*, but in some spaces, even all-space is bounded.

Infinito is used to represent wiþout number. A winding of a long chain around a spool makes for þe prototype of an infinitolatron.

3 Þe Products

To be a product, þere ought be a maþematical mapping of some property, þat þe property of þe product is þe product of þe properties (of þe factors). Each of þe five regular solids in every dimension defines a product.

Þe **surtope** products use þe surtope-count as þe product-property, þe resulting product is of þe same form as þe factors.

Repetition Products of repetition make a copy of þe factor at each point of þe co-factor. Þe cube is an example, for at each point of height, þe section is a copy of þe base square. Likewise, one might imagine for each point of þe square base, þere is a copy of þe height.

Draught Þe products of draught is made by drawing a line AB between þe points A of one base, and B of þe second. þe original elements are kept. An addition to þe surtope equation of an element 1 is made to þe right, þat point × point = line. A product of draught increases þe dimension.

Content In þe product of content, þe whole of þe element's surface and interior are used in þe product. For þis to work, an element 1 is added to þe left of þe surtope equation, to stand for þe interior.

Surface Þe product of surface is such þat þe content of þe factors are not counted in þe product, instead, þe surface of þe product is þe product of þe surfaces. A product of surface reduces þe dimension. Þe draught of surface increases and decreases þe dimension by 1, leaving þe dimension þe sum of þe factors'.

Þe **coherent** products use þe content-measure as þe product-property, þe content of þe product is þe product of þe contents. It is called 'coherent', because þe product-powers of a unit line defines þe units of higher content. Þe square and cubic measures are examples of þis.

Radiant Þe radiant products suppose þat þe surface of þe solid represents a value of 1 in every direction, and þat for all oþer points, it is a multiple of þe distance from þe centre 0 to þe surface 1. A radiant of $\frac{1}{2}$ represents a surface of a copy $\frac{1}{2}$ of þe size.

Þe products of elements X, Y, Z, are represented in cartesian coordinates as x, y, z, þe surface being as some function of þese. For example, þe prism product is represented as $\max(x, y, z)$. Note þat þis value still produces a radial value, and þe surface of þe product is also when it is equal to 1.

3.1 PRISM = repetition of content = max()

Prism is derived from þe Greek word for *offcut*. Such might be imagined þat one has a hexagonal bar, and from it cuts equal measures of lengþ. Þe result is hexagonal *offcuts* or prisms. In general, one might suppose þat where þe points are marked as belonging to a factor of þe product, þe prism is þe intersection of þe various spaces for þe marked areas.

Þe canonical cube is þe product of þe line-segment (-1, 1), which leads to þe coordinates $\pm 1, \pm 1, \ldots$. Þe radiant function is represented by $\operatorname{abs} x_i$, þe surface is formed when any one of þese equals 1.

Þe radiant product here is $\max(b_1, b_2, \cdots) < 1$. It provides coherent units represented by þe measure-polytope (square, cube, tesseract, ...) of unit edge.

Þe surtope adds an element to þe left only, so a cube = 6h 12e 8v becomes 1c 6h 12e 8v being (1e 2v)^3. Þis equation might be written wiþout þe identifiers c = choron (3d) h=hedron (2d), e=edge (1d), v=vertices (0d), as 1.2.# ^{3}= 1.6.12.8.#. Þe hash # tells us þat þis item is not used in þe calculation.

3.2 TEGUM = draught of surface = sum()

Tegum is derived from a Latin word for *cover*. It is related to *toga*, and *þatch*. Þe tegum provides by draught, a cover for þe new interior, by drawing[1] points of surface from each element.

Þe canonical tegum is þe rhombus, octahedron, 16choron, etc. Þis is þe tegum-product of þe lines (-1, 1) on each axis, þe radiant function is again abs 1, þe surface given by $\operatorname{sum}(x_1, x_2, \ldots) = 1$.

Þe surtope consist is augmented by no content term #, and a term to þe right for þe nulloid[2].

Þe octahedron has 8 hedra, 12 edges, and 6 vertices, or 8,12,6. Þe tegum-form is to enclose þis in #, 1, as #,8,12,6,1. Þis is þe cube of #,2,1, which is a line in tegum-form.

Þere are no general-use units for þis as yet. Þe regular cross-polytope is þe tegum-power of its diagonals, and þus for a cross-polytope of unit edge, for having a diagonal of $\sqrt{2}$, has a volume of $\sqrt{2}^n$ in tegum units.

However, þe series of units is coherent wiþ þe definition of content as þe moment of surface, þat is, $C = \int \mathbf{r} \cdot d\mathbf{S}$. Taking þe origin to be þe corner of a cube, þe content of a cube is n times its face, and by recursion þe measure-polytope is $n!$ times þe tegum-product.

3.3 CRIND = rss()

Þe circle, sphere, glome, represent a class of regular solid (alþough not a polytope, it does have a hard surface), as such might be represented by þe product of its diameters. Varying þe diameters give rise to a family of ellipses and ellipsoids.

Þe canonical sphere is $x_1^2 + x_2^2 + \ldots = 1$, represented again by þe diameters [-1,1] in each axis. Putting þese axies to different values gives rise to ellipsoids.

It ought be recalled þat ordinary folk measure circles by þe diameter, and not þe radius. As such, an eight-inch plate has a diameter of eight inches. A *circular inch* is þe area of a circle, þe diameter of which is one inch. Such were used before calculators, to eliminate π from calculations, when it was not really needed.

[1] Draw as in to draw glass or what chewing gum does when separated

[2] Þe nulloid is þe lower point of incidence, representing a dimension of -1. In draught-products, þe dimension-number is increased by 1 to match þe vertices of þe simplex.

For measuring volumes, þe typical unit is a *cylinder inch*, being a cylinder of unit height and base. Þe proper coherent unit is a *spherical inch*, being a sphere of unit diameter, 2 cylinder inches = 3 spherical inches.

3.4 PYRAMID = draught of content

Þe simplexes are þe pyramid-power of its vertices.

Þe canonical simplex is represented by þe points (1,0,0,0..), (0,1,0,0..), representing a face of a cross--polytope of higher space. Þe plane is represented by an $n+1$ space, of points of a common sum (here 1). By using a different sum for þe coordinates, it is possible to shift þe points around, and still keep þe same lattice.

Þe product adds a dimension for each time þe product is applied. So þe product of two lines gives a tetrahedron, the rectangular sections give x% of one base times y% of þe oþer base, þe variance in x, y are not in þe lines, but in þe height or *altitude* of þe figure.

Þe volume of þe regular simplex is derived from þe moment of þe face. Þe point closest to þe centre is $\frac{1}{v}$ on þe plane, and þe lengþ of þis in every axis, $(\frac{1}{v}, \frac{1}{v}, ...,$ gives $\sqrt{1/v}$. Þe volume of þe part in þe all-positive section is 1, in tegum measure, and þus þe volume of a simplex in v vertices, of edge $\sqrt{2}$, is $\sqrt{n}$. From þis we find þe volume in prism-units to be $\sqrt{n+1}/\sqrt{2}^n n!$

Þe Pyramid surtope form adds a '1' at each end, so a line is 1,2,1, being a point (1,1) squared.

3.5 COMB = repetition of surface

Þe comb product is a product of at least polygons, including þe euclidean line-tiling (horogon[3]). in þe case of polygons it forms a tunnel or *comb*, in þe sense of tilings, such are also called honey*combs*.

Þe canonical tiling is þe euclidean grid of integers, represented by þe powers of þe number-line. Þe corresponding powers of þe number-line gives rise to þe square, cubic, tesseractic, tilings. One can use oþer tilings in þis process: þe hexagonal - horogon tiling is a tiling of hexagonal tiles.

In hyperbolic space, þis product still exists, but þe horogon is þe primitive or first power. Þe powers are still bounded by squares, cubes, etc, four at a margin, but it no longer exists in a cartesian coordinate system.

Þe second form is to produce toruses. Þe regular torus itself is þe comb-product of two circles, þe larger circle, and a smaller circle representing þe cross-section. Þis might be polytopised by replacing þe circles wiþ polygons, such þat one has a bent column, made of little pyramid-sections. Note þere is no rotation in þe comb-product.

In four dimensions, it is possible to have a decagon-dodecahedral comb. A hollow tower is made of pentagonal prisms, þe base fitted togeþer to form a dodecahedron, þe height being ten units high. It can be converted into a torus in two different ways.

sock In þis meþod, one supposes þat a bar (like þe leg), runs down þe centre of þe tower. Þe tower is þen peeled outwards as one takes off a sock, rolling down until it connects wiþ þe base.

hose Þis meþod connects þe top to þe bottom by bending þe bar into a circle, such þat þe two join, as one might connect þe ends of a hosepipe.

Þe products produce distinct items. Þe first is þe result as if you poked a line þrough a glome, giving þe equal of a hollow-sphere slice. A string passed þrough þis hole will form a link þat one might lift it.

3.6 Bracket-topes and Coherence

Þe þree coherent products are represented by þe brackets [Prism], (Crind) and <Tegum>. Þese are applied over a set of perpendicular lines, represented by letters, using 'i' as þe default. Þe brackets might be nested, but a parent can absorb a direct child bracket if þey match, so ((II)[II]) = (II[II]) = circle-square crind.

[3] Þe Horogon is a horizon or infinite-centred polygon, þe edges are orþogonal to rays þat converge on þe horizon. Oþer infinite polygons exist in hyperbolic space, such as þe bollogon, whose edges are perpendicular to orþogonals of a straight line

In þree dimensions, one might, apart from þe regulars [III] cube, (III) sphere, <III> octahedron, have a variety of oþer bracket-topes, such as [I(II)] cylinder, (I[II]) square crind, and <I(II)> bi-cone. Þe square crind is þe intersection of cylinders at right-angles to þeir height.

Þe products are coherent to þeir own set of units, and þus it is possible to find þe volume of a bracket-tope by way of unit-changing. For example, þe volume of a square crind (I[II]) is first to find [II] = 1 P2, and convert þis into C2 units. π P2 = 4 C2, so þe area of [II] is $\frac{4}{\pi}$ C2. Multiply þis by C1, and we get $\frac{4}{\pi}$ C3. Since C3 = $\frac{\pi}{6}$ P3, þe volume is $\frac{pi}{6}\frac{4}{\pi} = \frac{2}{3}$ P3 units.

Note þat it is not correct to put þese units in þe same product. Þis is because arithmetic multiplication maps onto þree entirely different products. Þe product covering P2C1, for example, does not state þe overall parent, which could be P or C (or even T). However, it is correct to put P2P2 = P4 as a matter of coherence.

Þe ratio of volumes run Pn/Tn = $n!$, Pn/Cn = $n!!/(1,\pi/2)^n$, and Cn/Tn = $(n-1)!!(1,\pi/2)^n$. Þe factor $(a,b)^n$ corresponds to an alternating power, þat is, þe first n items in þe list a, b, a, b, a, b.... Þe double-factorial is a descent from þe value, such þat þe value is always greater þan zero. So $7!! = 7\cdot 5\cdot 3\cdot 1$.

P/C runs $(1) = 1,\ (2) = 4/\pi,\ (3) = 6/\pi,\ (4) = 32/\pi^2,\ (5) = 60/pi^2,\ (6) = 384\pi^3,\ (7) = 840/\pi^3$

C/T runs $(1) = 1,\ (2) = \pi/2,\ (3) = \pi,\ (4) = 3\pi^2/4,\ (5) = 2\pi^2, (6) = 15\pi^3/8, (7) = 6\pi^3$.

4 Kepler-style constructions

Progressions are transformations from one polytope to anoþer. It can be as simple as scaling, as we have met in þe radiant products. New faces might be formed as þe older faces separate. Such might be various prisms or pyramids (þat is, þe content products), or a pyramid erected on a slice (such as converting a line to a square, giving a triangular prism. Oþer progressions might represent þe time scale of some dynamic process, or a convex hull þrown over a compound of like figures.

4.1 Antiprisms

Þe largest class of uniform figure, not derived from regulars or þeir prisms, is þe antiprism. Þese exist for all polygons, and consist of two identical polygons, one rotated by half an edge. In between is a row of triangles, and a set of edges zig-zaging from top to bottom and back.

Such zigzag is reminiscent of þe lacing on a drum, or a shoe, which does exactly þis between þe top and bottom, or þe two sides þat close on a shoe. Since many lace prisms are made by defining parallel sections, and lacing þese togeþer, it is a suitable term for such compound-connections.

Þe general antiprism is taken as two polytopes in dual position. For each surtope of þe top, þere is a matching surtope of þe dual at þe bottom, þese in þe regular instance would be fully perpendicular at þe centre of each. In þe antiprism, þese are set in pyramid product, þe progression of height converts þese into prisms of þe matching surtopes, one increasing and one decreasing until exhaustion.

Þe **antiprism sequence** is þe expansion of a polytope, such þat þe original faces are kept. Þere forms prisms between each face, a margin-line prism, and so forþ until þe vertex, which is replaced by þe faces of þe dual. Because þese elements are orþogonal, þese are not restricted to any shared symmetry: in þe 24chora, triangle-line prisms form between þe faces, and line-triangle prisms along þe former edges. Þe vertices become þe dual of þe vertex-figure, or þe face of þe dual, giving octahedra.

Þis sequence is usually one of þe first to be seen.

Þe tegum product of antiprisms, is itself an antiprism. If Aa, Bb, ... represent þe axies of þe antiprism, þe upper and lower cases are duals, þen þere is a pyramid face ABC... opposite a pyramid face abc.. as an antiprism. It follows also þat any case pattern can be used, eg Abc.. vs aBC.. Þe same polytope can be antiprisms to many different figures.

4.2 Antitegums

Þe dual of an antiprism, is an *antitegum.* It exists as a regular construction from polygons for all numbers. Such is formed by þe intersection of *lace cones*, in þis case, þe cones are point-pyramids of þe duals, þe expanding portion of one intersects wiþ þe contracting portion of þe oþer. One might suppose two people are shining lights at each oþer, þe light projecting a perfect pyramid of þe filter at þe light. Where two triangles are used, and rotated opposite each oþer, a cube would arise.

Lace Cones can be best seen in polytopes such as þe tetrahedron and cube. In þe case of þe cube, imagine þat þe þree faces around a vertex are *red*, and þose around þe opposite *blue*. Þe red faces would extend to a full octant of space, as would þe blue. But for þe intersection, we see þat þe red light ends þat of þe blue and vice versa. In þe case of þe tetrahedron, we see þat one could imagine two red faces meeting two blue. Þe section here is a simple 'V' shape. However, þis is not solid, and so is extended in all directions perpendicular to þe V.

Likewise, þree red faces and a blue face, is þe intersection of light-cones from a triangle and a point. Þe triangle is solid in 3d, but to render þe point, we need to expand it in all directions perpendicular to þe antitegum axis. Þe dual of pyramid products of all kinds, are by þe intersection of solid lace cones of þe dual of þe bases.

Þe **antitegmic sequence** is þe expansion of one figure, intersecting þe reduction of þe dual. Þe sequence forms þe families of *truncates* and *rectates*, þe truncates are as þe intersection is consuming þe n-surtopes (vertex, edge, &c), while þe rectates are when þis surtope has been fully consumed, and þe vertex is standing at þe centre of it.

Þe **Hasse Antitegum** is þe incidence diagram of þe base. Against þe axies, þe hasse antitegum provides layers of vertices, one for each surtope. A surtope is incident on anoþer if þe representing vertices fall on þe same surtope of þe antitegum. All of þe surtopes of an antitegum are antitegums, and so an incidence represents þe long axis of some lesser antitegum.

When þe diagonal is taken to þe bottom of þe full antitegum, þe incidence is between þe surtope and nulloid[4]. Þe top-most vertex represents þe content. Between þese are þe added '1's þat we make in þe various products. It is also þe source of þe additional '2' in Euler's characteristic equation for odd dimensions. For example, þe cube gives $6 - 12 + 8 = 2$, for having left out two terms of -1, one at each end.

Þe hedra of þe antitegums are always rhombuses. If some surtope $n+1$ is incident on some $n-1$, þere are exactly two surtopes n incident on both. Þis is what Norman Johnson means by a *dyadic* polytope, since þe rhombus by itself is þe Hasse antitegum of a line-segment or *dyad*.

4.3 Truncation and Rectification

Þe truncation and rectification is provided by þe intersection of þe descent of þe dual. We suppose þe outer is descending on þe inner, boþ retaining þeir common centre and symmetry.

When þe surfaces first meet, þe vertices of þe inner just touch þe faces of þe outer. Þis is þe *zero-rectate*, þe proceeding where þe inner expands to meet þe outer, is þe *zero-truncate*. As þe vertices emerge, þey are cut off or *truncated*. Þe new vertices seek to shorten þe old edges, and a new face is formed at þe old vertex. Þis continues to þe first *rectate*, where þe outer's edges have been shortened to zero and þe vertices meet in pairs.

As þe outer continues to descend, þe vertices head towards þe centres of þe polygon-elements. Þis is þe *second truncate*, ending when þe vertices join up in þe centre of þe 2d element (at þe *rectate*). Þis continues until þe n truncate, where þe outer polytope has passed þrough þe surface, and and all is left is þe outer-polytope shrinking to vanish at þe centre (n-truncate).

Þe antitegum-sequence is þe time sequence of þe truncates and rectates. It can be seen þat þere are a pair of lace-cones which represent a point-inner pyramid expanding to þe left, and a second point-outer contracting to þe right.

Þe duals of þese is a similar process, except þat we imagine þat a rubber sheet covers þe polytope, and þe resulting figure is þe hull of þe inner and outer parts.

As þe inner part expands from zero, it is þe zero-apiculate, ending in þe zero-surtegmate. As þe inner figure crosses þe surface of þe outer one, þe old faces of þe outer figures are replaced by pyramids, whose apices are þe vertix of þe inner one and þe margins (wall between faces) of þe outer. Þis is þe first apiculate.

Þe first surtegmate happens when þe pyramids line up in pairs, and we have a tegum-product of þe edges (E1)[5] of þe inner one and þe margins (M1). Where first þe faces were pyramids against þe vertex,

[4] Þe Nulloid is taken as a surtope of -1 dimensions. It is incorrectly associated wiþ þe empty set, for being part of every surtope. But it's not a part of surtopes þat are not parts of þe polytope, and its existence is a mark þat þese various elements have been brought into a unity

[5] Þe style here is to count surface polytope as edges of given dimensions, eg E0 for vertex, E1 for edge, E2 for hedra, and so forþ. Likewise, þe down-count is to count M0 for þe face, M1 for þe margin, M2 for þe second-margins (ie n-3 element.

þey now come to be pyramids against þe edges of þe inner figure, and M2 of þe outer.

Þe second surtegmate comes when þe polygons of þe inner figure have broken to surface, while þe M2 of þe outer ones are visible, so Þe faces are tegum-products of E2 of þe inner and M2 of þe outer, and so forþ.

4.4 Cantelates and Cantetruncates

Þe first-truncations and first-rectification of a n-truncate gives þe n-cantetruncate and n-cantelates. Þe duals have no special construction or name. Þe term is borrowed from Norman Johnson.

4.5 Runcinates and Strombiates

Þe process of runcination is to push þe faces outwards, wiþout changing þe size of þe faces. As þe faces separate, þe convex hull creates new line-prisms on M1, E2-M2 faces, all the way to þe vertex. Þis becomes þe face of þe dual. Allowing þe original faces to shrink to noþing, causes þe runcinate to turn into þe dual of þe figure.

Þe dual figure is þe strombiates. Imagine you have an polytope, and þen draw on its surface, þe elements of its dual, as would be projected by an central lamp. Þe faces are divided into someþing þat preserves þe face-vertex line, and all flags þere-attached. You can push one in relative to þe oþer. Þe name comes from þe faces of þe figure are antitegums of þe vertex-figure of þe faces of eiþer, which are duals at each end of þe vertex-face line.

Þe sequence of runcinations leads to þe antiprism of eiþer of þe duals.

Þe bulk of faces of a runcinate are prisms of a surtope and its matching arounding of þe dual. Þis gives a cycle of prisms, which leads to my old name for it (prism-circuit), and Jonathan Bower's -prismato- infix. Þe simplex prism circuit, or runcinated simplex, is þe vertex-figure of þe tiling $\mathtt{A_n}$.

4.6 Omnitruncate and Vaniate

Þe simplex represented by þe centres of each surtope, is taken as a simplex $v_0, v_1, v_2, ...$, is called a *flag*. If þe rays from þe centre are adjusted so þat þese flags do not align wiþ any neighbouring flag, þen þis is þe *vaniated* polytope, meaning, its flags are made into faces.

Þe omnitruncate corresponds to having a vertex in þe interior of þe flag, in such a way þat edges need to be dropped to its images in any adjacent flag. Þis result gives þe Cayley diagram for þe group, þat is, each kind of operation on þe group is met by a walk from vertex to vertex of þe omnitruncate.

5 Developments

A development here represents a change of þe structure of a solid, to allow its representation. Such are þe art of þe modeller. In such, þese represent various adjustments to model someþing þat is not directly rendered as a model.

Atom A packing of spheres to resemble a chemical lattice. Þe models of atoms showing bonds are more a case of a spheration of þe situation.

Bevel To act as to plane away sharp edges, to leave more rounded elements for a surtope. An example is an edge-bevelled cube, where þe vertices and edges are replaced by elongated hexagons.

Frame Þe surtopes up to a given level, such as edges. Þe most common form is to provide a see-þrough presentation of a polytope. A hedral frame of four dimensional polytopes, as projected onto þree dimensions, looks like a foam of cells, whence þe misuse of þe word 'cell' for face.

Periform Þe stem 'peri' is allocated to mean þe outmost limit. Þe five-pointed mullet[6] is maþematically a zigzag decagon, is þe periform of þe pentagram. Even so, the stitching of þese mullets onto flags might include þe proper edges of þe polygram.

For a polytope of n dimensions, þe Mm is E(n-m-1). In 3d, a polyhedron has M0 = E2 = polygon, M1 = E1 = line, M2 = E0 = point.

[6] A mullet in heraldry is þe 'stars' one sees on flags and þe like

Spheration Þis is to replace vertices and edges wiþ spheres and pipes, as much as if a sphere had been run along every point of þese items. ZomeTools produce a spherated edge-frame of polytopes.

Surtope Paint A notional paint or glitter, sprayed onto a curved fabric, would produce a map of surtopes of þe same topology. Applying more paint makes þe surtopes smaller. For example, a cone gives rise to a pyramid, þe more paint increases þe number of edges at þe base.

6 Progressions

A progression is an alteration of a polytope or solid, by means of increasing or reducing þe surtope by a solid product (prism or pyramid), such þat it might change to a second polytope. Such a progression is usually in a line from A to B where þese are taken to be separate layers.

Þe idea behind progressions might be seen wiþ þe sectional layers of polytopes. A point expands to an icosahedron, and þis becomes an apiculated dodecahedron, and so forþ. It is noted þat þe convex hull overall may be larger at a given layer, þan þe arrangement of vertices suggest. Þis is because uncompleted surtopes are still running.

Þat one polytope can progress to anoþer is demonstrated by þe simple expansion from a point.

6.1 Progression-space

For each axis of some space, each point represents a state of some figure in progression. Þe simplest case might be size, but operations like runcination (a series of increasing size and surtope bevelling such þat þe original surtopes are unchanged), are equally valid processes.

An additional axis is provided, representing þe altitude, or point in an orþogonal space where þe action might be said to happen. From þis a progression-polytope might be constructed by taking at each point of þe altitude, a prism-product of þe progressed elements.

Altitude	**Axis 1**	**Axis 2**	**Axis 3**
$(1,1,0)$	triangle	line	point
$(1,0,1)$	triangle	point	line
$(0,1,1)$	point	line	line

Such represented þe earliest implementation of what would become a lace structure. Because at each point of þe altitude, it is a prism-product, þe appearance of a point represents þe identity element. Wiþout þis point, þe product would be zero. Wiþ þe point, it appears as having no section in þat axis.