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WALLS AND BRIDGES: THE VIEW FROM SIX DIMENSIONS

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Abstract: *Walls divide, bridges unite. This idea is applied to devising a vocabulary suited for the study of higher dimensions. Points are connected, solids divided. In higher dimensions, there are many more products and concepts visible. The four polytope products (prism, tegum, pyramid and comb), lacing and semiate figures, laminates are all discussed. Many of these become distinct in four to six dimensions.*

1. WALLS AND BRIDGES

Consider a knife. Its main action is to divide solids into pieces. This is done by a sweeping action, although the presence of solid materials might make the sweep a little less graceful. What might a knife look like in four dimensions. A knife would sweep a three-dimensional space, and thus the blade is two-dimensional. The purpose of the knife is to divide, and therefore its dimension is fixed by what it divides.

Walls divide, bridges unite. When things are thought about in the higher dimensions, the dividing or uniting nature of it is more important than its innate dimensionality. A six-dimensional blade has four dimensions, since its sweep must make five dimensions.

There are many idioms that suggest the role of an edge or line is to divide. This most often happens when the referent dimension is the two-dimensional ground, but the edge of a knife makes for a three-dimensional referent. A line in the sand, a deadline, and to the edge, all suggest boundaries of two-dimensional areas, where the line or edge divides. We saw above, the sweep of an edge divides a solid.

In the proposed terminology, the *margin* takes on the role of a dividing edge. Face and surface suggests a bounding nature, and so are taken to refer to containing a solid: a four-dimensional face has three dimensions. A margin angle is the term that replaces the *dihedral angle*. In four dimensions, dihedral angle is about as relevant as a corner angle in three dimensions.

The decision to use the walls and bridges notion is more that certain words have acquired powerful meanings that may lead to confusion in higher dimensions. It is probably more important to keep the dividing nature than the two-dimensionality, of a plane or a face. But I do consider later on the style of why a dimension-based terminology is also important to keep.

The vertex-edge and face-margins are topological duals in every way. Where one can do things in one, there is a corresponding dual for the other. Among the mathematicians, the vertex-edge set makes for the simplest constructions: all vertices are essentially alike, and edges have only length. For those who study crystallography, the face-margin set appears to make the greater sense. Many of the crystals occur in the shapes of Catalan figures.

Polytopes carry the names referring to their faces. Yet we deal with vertices and edges. In any case, there is an asymmetry of names that needs to be corrected. My endeavours into this field have been largely to address this asymmetry, largely by filling in the holes.

The starry polytopes are made by face-extension. However, the usual process of finding them is by creating new faces in existing vertices of the dual. While the two are the same process, the process is converted from a face-centric process to a vertex-centric one. Faceting and stellation are dual processes. With faceting, we keep the same vertices and span new edges, …, faces. With stellation, we keep the face planes, and find new margins ,…, vertices. The faceting or stellation is regarded as less extreme when a greater number of elements are kept. For example, a first-margin faceting keeps all the margins, and span new faces. A great dodecahedron is a first-margin faceting of the icosahedron. The dual is that a stellated dodecahedron is a first-edge stellation of the dodecahedron: a complete dual in every way.

The method used by Jonathan Bowers in his program to discover the uniform polychora is to use first-edge facetings. In essence, an army consists of all the polytopes that have the same vertices. This divides into regiments, which share the same vertices and edges. This descends into companies, and so forth. The process corresponds to vertex-facetings, first-edge facetings, and so forth. A corresponding dual would be to have face stellations, first-margin stellations, and so forth, forming a navy of polytopes. Stellations are more complex, because unlike vertices, faces do change.

We can talk of inner stellations, or outer facetings. An inner faceting has the same face-planes as the figure, but lies inside it. The innermost stellation is the core. Likewise, the outermost faceting is the hull. The core and hull are both convex.

The polytopes that form a first-edge faceting or regiment share a common set of vertices and edges. One can talk of a first-edge subfaceting, where the vertices and edges form a subset of the original set. A pentagonal antiprism is a first-edge sub-faceting of the icosahedron. Just as with a set being contained in a superset, one can also talk of super-facetings and super-stellations. An icosahedron is a superfaceting of the pentagonal antiprism: it has all the vertices of the former, and two additional ones.

1.1 The Rhombus

A thread is a sequence of polytopes, one from each dimension, that share some common property. The classic example is *line, square, cube, tesseract, …* These form a sequence of measure polytopes. But threads can cross and converge, especially in the lower dimensions.

The rhombus is a relatively useful polygon. It has four equal sides, and a pair of axies bisecting at right angles. What ought be the polyhedron that should inherit the spirit of the rhombus in three dimensions. I have three different words for the three different qualities that the rhombus gives, all based on the stem *tegum*.

A *rhombohedron* is a figure that continues the equal sides of the rhombus. Where in two dimensions, it is a square, stretched on its long diagonal, in three dimensions the cube gets stretched in a like manner. While the rhombohedron tiles space, and is useful in crystallography, the general class it belongs to I call *antitegums*. The rhombohedron is a triangular antitegum. Antitegums are the dual of antiprisms.

A second element is the notion of crossing axies. In three dimensions, one can have three crossing axies, giving rise to a kind of isoface octahedron. This is the dual of the general rectangular prism. We might easily call it a rhombic octahedron, but the name selected for this is *tegum*. The dual of any prism product is a tegum product of the duals.

Kepler named a number of uniform figures and their duals with the name *rhombo-*, eg rhombocuboctahedron. When we consider there is no rhombus in three dimensions, we might ask which of the above two meanings is meant. The rhombic dodecahedron tells us what is going on. The diagonals of its rhombic faces are the edges of the cube and octahedron. In four dimensions, the edges of the figure cross the margins or polygons of the dual. The resulting faces would be a tegum product of matching dual elements. This gives a *surtegum*, or surface-tegum figure.

1.2 The view from six dimensions

The terminology I have selected for higher dimensions is tested in six dimensions. Many of the different threads become quite distinct in six dimensions. Also, in six to eight dimensions, there is a fascinating series of polytopes discovered by Th. Gosset. I could have set the thing up as *the view from eight dimensions*, but I can't visualise that many dimensions.

Many useful distinctions become more apparent in six dimensions. This is because there are a larger number of intermediate dimensions. There are four different products, one for each of the infinite regular polytopes. Of these four, two are distinct in four dimensions, and the other two have to wait until five dimensions to become distinct. In three dimensions, we can largely ignore these.

At the moment, I am writing a polytope glossary, called the *Polygloss*. This sets down a large vocabulary where the terms are largely defined to be consistent across the higher dimensions. It only goes as far as eight dimensions, but the general pattern is there.

1.3 Around and Surround

In the higher dimensions, these terms are used to define quite distinct meanings. Consider a three-dimensional subspace in six dimensions. A figure that is solid in the subspace is both surrounded and arounded by different kinds of spaces.

Surrounding happens in the space that the thing is in. When one surrounds a fort, one creates a barrier to ground transport to it.

Arounding happens in the space perpendicular the surface. When one dances around the maypole, the dance is in a circle that encloses, but does not include, the maypole. The maypole is a one-dimensional affair, but the dance happens in a two-dimensional space that crosses it at one point: the ground.

The prefixes chosen to suggest surround and around are *sur-* and *ortho-*. So things that have *sur-* in them happen in the space bounding a figure, and *ortho-* suggests a space entirely perpendicular to it.

In and out happen across a surface. Any shape that has a boundary potentially has an inside and an outside. An enclosure made on the ground exists essentially in the plane of the ground, and therefore has an inside and an outside. However, birds on the wing would not observe this particular distinction. A shape drawn on a four-dimensional plane of six dimensions has an interior and an exterior in very much the same way as when it is solid in four dimensions.

1.3 Hyperspace, Slabland, and many products

Hyperspace means space over solid. It is useful to assume higher dimensions, some mathematical theorems rely on this assumption. But to apply it to four dimensions or any other particular dimension would lessen this utility. Calling a tesseract a hypercube is like calling a square a hyperline.

Slabland is an approach to higher dimensions. One imagines that in a two-dimensional world acquires thickness, like a pancake. The cartoon character Gumby, who resembles a man cut out of a layer of green foam sheeting, would not look out of place here. Slabland is a useful concept because we can make the transition from one dimension to another by inventing thickness to interact. The more common form is Filmland, where we are paper-thin film characters that blow around in a higher dimension.

The Slabland idea is also important because it can convert uniform polytopes into slab prisms in the higher dimension: a hexagon becomes a hexagonal prism. One could then have the same sequence number applied both to the polytope and its prism. In my series, I give the number 7 to a dodecahedron, and 7 to its four-dimensional prism. The first in the series is the ultimate slabland device: the square, cube, tesseract, &c.

Slabland gives way to the Cartesian product. The word *prism* means *offcut*, such as one might cut off a length of wood. Imagine cutting a hexagonal pole into hexagonal prisms. In terms of three dimensions, one can regard a hexagonal prism as being a hexagonal offcut from a layer, or a short height off a long column, or the common intersection. In terms of co-ordinates, a hexagonal prism projects onto a hexagon in two dimensions, and its height into the third.

In four dimensions, the prism product becomes distinct. What this means, is that there are prisms that do not come from Slabland. One could place a hexagon in two dimensions, and a pentagon in the other two, and consider their common intersection.

Another product that becomes distinct in four dimensions is the *tegum* product. This makes the duals of prisms, but has its own identity. The original word proposed for it was (tent), but somehow *tabernacle* is already used. Tegum means *to cover*. The sense is that the surface of a tegum covers its axies like a tent covers its pegs.

The land of tegums is Bouyland. The shapes of the previous dimensions are converted into bipyramids that float around the surface like bouys at sea. A hexagon becomes a hexagonal bipyramid or tegum. The first shape of bouyland is the square, octahedron, 16-choron, &c

To make a distinct tegum, we need to find something that distinct from Bouyland. This is done by replacing squares or higher with some other figure from the same dimension. Replacing a square in the octahedron by a pentagon makes the octahedron into a pentagonal bipyramid or pentagonal tegum. A 16-choron, taken as the product of two squares, can become a pentagon-hexagon tegum, with a pentagon in one pair of axial dimensions, and a hexagon in the other two. The surface consists of thirty disphenoid tetrahedra.

Tegums can be used as a measure unit also. The ratio of a tegum unit to the prism unit is in the ratio of one to the factorial of the dimension. In five dimensions, the prism unit is 120 times greater than the tegum unit. A tritegmal foot refers to the volume of an octahedron, the diameter of which is a foot. The solid angle of a simplex, measured in tegmal radians, gives a value between one and the square root of N/e.

Fireland makes a shape into pyramids. Our hexagon becomes a hexagonal pyramid. The first member of these is a series point, line, triangle, tetrahedron, pentachoron, … The first distinct pyramids are found in five dimensions. This is where we can replace pairs of triangles of the hexateron with other polygons. Unlike prisms and tegums, the pyramid adds a dimension for every application: this becomes part of the height. So where the tegum and prisms are the products of lines (diagonals or edges), the pyramids are a product of points (apexes or vertices).

In five dimensions, we have the hexateron being seen as a triangle triangle pyramid, and we can replace the triangles by any other polygon. We could have, for example, a pentagon hexagon pyramid. A slice through the altitude gives rise to a pentagon hexagon prism. When the thing is projected onto four dimensions down the height, the result is a pentagon hexagon tegum.

The last land is Layerland. This does not apply to polytopes but to Euclidean tilings, and by extension, to horotopes. The way this land works, is that it replaces a hexagonal tiling by a whole stack of layers of hexagonal prisms. The first member is a member of tilings of measure polytopes: quartics, cubics, tesseractics. A tiling of squares is a three-dimensional polytope, acting in the role of a two-dimensional honeycomb.

The comb product is the general product for layerland. The first comb-products that don't come from layer-land are five-dimensional polytopes, which appear as four-dimensional tilings. In this, we treat the tesseractic as the comb product of two quartics (square tilings), and replace each by other two-dimensional tilings. One could have a tiling of triangle-hexagon prisms, or a trilat hexlat comb.

In hyperbolic space, the members of layerland do not appear as tilings but as polytopes with a proper curvature, and a non-planar margin-angle. However, the comb product still applies. In hyperbolic space, the trilat {3,6} is a three-dimensional polyhedron, albeit with infinite radius. The comb-product {3,6}{6,3} gives rise to a five-dimensional polytope: that is, it looses a dimension.

One can do comb-products over polygons as well. This gives rise to only the Cartesian product of the surface. Where a pentagon-hexagon prism has eleven polyhedral faces, the corresponding comb is just the mat of thirty squares that divide the pentagon prisms from the hexagon prisms.

Circles and spheres can participate in all of the above products. For example, a cylinder is a circular prism. One can talk of bi-circular prisms and tegums, or a glomohedral prism (a 3d sphere × line prism).

The four products described above give rise to a rather attractive over-all symmetry. Adding a '1' to various ends of the surtope equation of the four classes of regular products converts these into power expressions. The same pattern makes for the generalised product. For example, a tetrahedron has 4 faces, 6 edges, 4 vertices. Adding 1 to each end makes 1,4,6,4,1 or 1,1 to the fourth power. We see that if we add ones to both ends of any polytope before multiplying, we get the consist of the product. For example, a square is 1,4,4,1 (4 edges, 4 vertices), and a point is 1,1. The product is 1,5,8,5,1. The square pyramid has 5 faces, 8 edges and 5 vertices.

The family of cubes or measure polytopes are powers of 1,2, the prism product adds a 1 only to the front of the sequence. A pentagon prism is 1,2 × 1,5,5 or 1,7,15,10. It has 7 faces, 15 edges and 10 vertices. Measure products preserve vertex-uniformity. That is, if two figures are vertex-uniform, so is the product.

The cross polytopes are powers of 2,1. The tegum adds only to the end of the product. A pentagon tegum is the product of 2,1 and 5,5,1. This gives 10,15,7,1. This has 10 faces, 15 edges and 7 vertices. The tegum product preserves the face-uniformity. That is, the product of two iso-face polytopes, like the Catalans or the Platonics, give rise to another isoface figure.

The family of quartics, cubics &c are powers of 1,1. Here the 1,1 represents an infinite sided polygon, and adding 1 to either end is not going to make any change. The numbers are proportional, in any case. None the same, the pentagon-hexagon comb is the product of 5,5 and 6,6, giving 30,60,30. This comb product is a mat of squares in four dimensions, with 30 faces, 60 edges and 30 vertices.

1.4 Polytopes and Mounting

A dodecahedron has twelve faces. There are many different kinds of dodecahedra, all of which are bounded by twelve faces. The sense of -hedron is then a mounted polygon. This particular notion has been preserved into the higher dimensions. The stem is derived from a Greek word meaning seat: it occurs also in cat*hedra*l church, meaning the church with the overseeing, or bishop's, seat.

The idea has been progressively extended into higher dimensions. A -choron is a mounted polyhedron. The sequence continues to 3d choron, 4d teron, 5d peton, 6d exon, 7d zetton, and 8d yottons. The names from four to eight dimensions are borrowed from metric prefixes: these are meant to stand beside numbers without confusion.

A *surtope* is a surface polytope, or polytope mounted on the surface. Just as polytope generalises the series point, line, polygon, polyhedron, … the surtope generalises the sequence vertex, edge, …, margin, face, cell. A cell is a solid surtope, such as a tiling might have.

When a polytope is mounted onto a second polytope, they share the interior of some surtope. When this happens, the two must also share the surtopes of the shared interior. That is, you can't mount polyhedra by placing the square face of one onto the triangular face of another. The join must match in shape and size.

The term polytope tends to get overused, more because there are not names for things that are not polytopes. It is as important to consider these as well. The style selected for the Polygloss is to use the concept of 'polytopes mounted with some result'. These are done by a series of Latin-and-Greek stems. We have already seen the stems meaning the likes of "mounted 4d polytope". We now look at the effects.

A *polysurtope* means many surtopes. It is a collection of mounted polytopes without any sort of definite aim. These might be used in topological maps, for example. If every surtope belongs to a polytope of the same dimension, one might call it a polysurhedron. A polyface is a thing made out of bounding polytopes: for example, a net or partially made model is a polyface. A polycell is several solid polytopes connected together.

An *orthosurtope* means the surtope that is orthogonal. The term is applied to the surtope of the dual, drawn in the space around, or orthogonal to, the original surtope. The dual of the orthosurtope is the surtope figure, a concept that generalises the vertex figure. This is topologically the same as the intersection of the surface with the orthosphere.

An edge-rectified polytope has its vertices in the centres of the edges of the polytope it rectifies. A cuboctahedron is an edge-rectified octahedron. The dual of rectification is surtegmation. An edge-surtegmated octahedron would create new faces, that are the tegum-product of the edges of the octahedron, and the margins of the cube.

A *polytope* means *many mounted polytopes*. There is no consistent rule for it, but the sense is some kind of closure, either a volume or margin completion. Different authors have definitions for it. In any case, it is hoped a wealth of new words might provide alternatives, and let *polytope* find a proper home.

An *apeirotope* means 'mounted polytopes without end'. The sense taken here is that the polytopes cover all of a space where they are solid. A tiling of hexagons, covering all of two dimensions, would be an apeirohedron.

An apeirotope can be treated as the surface of a hyperspace polytope. The faces of this hypertope become the cells of the apeirotope. Margins become walls. The hypersurface becomes a surcell.

A *planotope* has plane-mounted polytopes. While this is essentially the same as an apeirotope, it also has a volume. A tiling of hexagons and the half of all space it divides makes a planohedron.

An *anglutope* is a 'mounted polytope as a corner'. A single vertex of a dodecahedron appears as three different corners, one for each pentagon. The idea of anglutopes generalise this. It works in both directions: a pentagon has five corners, and a vertex has three pentagon-corners. Anglutope conveys the sense of *incidence*, or surtopes belonging to surtopes. A vertex may have incident faces, and such faces would be described as the vertex's *anglufaces*. One might call an incidence matrix an anglutope matrix, with columns representing the surtopes, and the rows representing the incident angulotopes.

A *horotope* is polytopes mounted on a horosphere or sphere that has an infinite radius. In Euclidean geometry, this is a flat surface. In hyperbolic geometry, this is a kind of sub-space that has Euclidean geometry. A tiling of hexagons, three to a corner, would form a horohedron. The term horotope is used to convey the sense of Euclidean surface geometry in both Euclidean and Hyperbolic geometries. A horosurtope is a surtope that is centred on a horopoint, or point on the horizon.

A *bollotope* is a polytope that follows a bollosphere, or hyperbolic radius sphere. A bollosphere is also called a pseudosphere or equidistant curve. The stem *bollo-* is derived from hyper*boli*c, in much the same way that *bus* comes from omni*bus*. Pseudo means false. It already has active use in this meaning, and it does not well to overload it with the sense of hyperbolic. An equidistant curve is just a curve equidistant from a straight line. A line of latitude is also an equidistant curve: it is equidistant from a straight equator.

A *glomotope* is a polytope mounted to make a globe. What this does is makes a single face wrap around to form a sphere. A glomohedron is the shape we call in 3d a sphere. In higher dimensions, there are 4-spheres or glomochora, 5-spheres or glomotera, and so on. *Sphere* can then refer to a solid sphere. The glomotopes participate in all of the polytope products. Even though some do not hold them to be polytopes, it is useful to treat them as polytopes just the same. They even have their own Schlaffli symbol allocated. A circle is {O}, a sphere is {O,O} and so on. A cylinder would be {}{O}, or a circular prism. When a Wythoff style construction is applied, this translates to shortening the axis. A prolate ellipsoid would become {;O,O;}, meaning the first two axies are equal, and shorter than the third, while an oblate ellipsoid is {;O;O}, where the first axis is shorter than the other equal pair.

1.5 Wythoff, Stott and Dynkin

Wythoff and Mrs Stott are both associated with discovering the great bulk of uniform polytopes, more by fait of having the right notion, and filling in the holes. The magic lies in the notions.

Wythoff relied on mirror-edge polytopes, and semiates to fill in the snubs. A mirror-edge polytope is one where the ends of every edge are images of each other in a bisecting mirror. The interesting thing is that edges do not have to be equal: every rectangular prism is a mirror-edged figure.

Given a mirror symmetry group, one can move the vertex around in the kaleidoscope, and look at the resulting figure. In three dimensions, the kaleidoscope has three sides, three corners and the interior. This gives up to seven mirror-edge figures for each group. These seven are completely realised in the icosahedral and octahedral groups, but the tetrahedral gives only two.

Mrs Stott's construction consists of moving surtopes inwards and outwards. This has the potential to create new faces. Imagine a cube covered by an elastic skin. If we grab the faces, and pull them out (keeping the same size), the old edges and vertices will give rise to new faces. We can do the same with any combination of vertex, edge and face, to give rise to seven figures per core figure, as before.

Combining the two gives rise to a fascinating idea. Consider the mirror-group as some kind of bounding plane, rather like an octant of the Cartesian system. This is in fact, the case for the group {2,2}. When we move a point around, it moves around in all of the other 'octants' as well, as if reflected in the walls. Mrs Stott's construction corresponds to moving the vertex parallel to an axis. The resulting axial system can be treated as a coordinate system, and the vertex as the apex of a position-vector.

The coordinates are set, so that a unit along an axis corresponds to unit elevation off the opposite face: this makes the points like (1,0,1) correspond to a mirror-edge polytope of edge 2. The length of this vector corresponds to the circum-diameter of the unit-edged figure.

In a sloping axis system, the way one finds the length is to use a matrix-dot. This is done in the same way as a dot product, but one of the two vectors is pre-multiplied by a matrix. The matrix used for this calculation is the Stott matrix, of which we shall comment further. Stott matricies can be used for hyperbolic groups as well, this will continue to give the edge of the resulting hyperbolic tiling. The value given is 2sinh(R/2L), where R is the radius of space, and L is the true length of the edge.

Dynkin's contribution was to provide a multi-dimensional notation for Wythoff's mirror-edge construction. The much-used Wythoff symbol assumes that a mirror is opposite an angle, a feature not replicated in higher dimensions.

The Dynkin symbol is a graphical affair, not suited for use in running text. It is very useful for the higher dimensions. One of the first things I did with it is to set it to running text, and greatly extend the versatility of it.

We can construct the Dynkin symbol in terms of a matrix. The diagonal elements of the matrix are set to 2, while the value for D(i,j) is -2cos(ij). The product of the Dynkin and Stott matrix is 2 I.

The Dynkin symbol represents mirrors by points (nodes), and the angles between these mirrors by edges (branches). Branches are only drawn if the angle between them is something other than a right-angle. The most common drawn branch is a '3' branch: the convention is that drawn unmarked branches reflect at 60°.

For a regular figure, the Dynkin symbol is a chain. This is easy to represent in text, because a chain can be made to lie down. For example, @--3--o--5--o would represent an icosahedron. But the dashes are entirely superfluous, and one could write @3o5o or, x3o5o. Since this also corresponds closely to the Schlafli symbol, one could write {;3,5,}.

Not all of the groups derive from regular polytopes. The way around this is to make the symbol represent a 'trace', or pseudo-regular figure. This is done by making some branches connect to a node further back or further ahead. In oGoEo3x3oAoBoCo, all of the branches connect the outer o node to the x node. The B branch is a 'third-subject node'. A branch connects a subject to an object. The subject of the B branch is x, the object is the o following the B. Since the x node is three back, it is a third-subject. These branches suffice to discuss all the hyperbolic groups where the simplex has finite content.

The special node z is used to indicate a return to the front of the chain. In the trace, it is still counted separately for the counting to find the subjects and object nodes. A group A5, represented by a pentagon of branches, might be written as o3o3o3o3o3z. In the Schlafli symbol, it appears as a colon, eg {3,3,3,3,3:}.

In the interests of symmetry, a *mirror-margin* figure is one where every margin lies in a mirror-plane. This is represented by the m node. Where an m node appears, the wall of the kaleidoscope is part of some margin. The neat feature is that one can dualise by swapping x and m. A cuboctahedron is o3x4o, the dual is a rhombic dodecahedron o3m4o.

Although figures can be both mirror-edged and mirror-margined, the correct style is to show only one. A cube is both x4o3o and o4o3m, but not x4o3m. The reason for this is that when applied across the direct product &, the x node implies a prism product, and the m implies a tegum product. So x4o3o&x5o is a prism product of a cube and a pentagon, while o4o3m&o5m is the same cube and pentagon in tegum product.

Circles and spheres can be treated in the same way as well. A circle is xOo, the higher dimensions effected by adding further Oo segments. So an xOoOoOo looks like a polychoron, and has O segments, so must be a glomochoron, or 4-dimensional sphere. Something like xOo&x is a circular prism, or cylinder. In four dimensions, we can have xOo&x5o, a circle-pentagon prism.

1.6 Laceland – Antiprisms and Antitegums

Kepler described among the uniform figures an infinite family of figures called antiprisms. These are a kind of prism, where the edge of one base corresponds to the vertex of the other. Triangles, not squares, form the sides. From higher dimensions, two important threads pass through here. One of these makes the pentagonal antiprism into a semiated decagonal prism: that is, what one gets by removing alternating vertices of a decagonal prism. Semiation splits further into finer threads, so it is useful to deal with semiates by new names.

An antiprism resembles some kind of drum, where the top and bottom are tied together with lacing. In higher dimensions, the name of antiprism is allocated to a similar kind of prism-like thing where the top and bottom bases are duals. The side faces are pyramid products of surtopes and the corresponding orthosurtope.

The idea of different-style top and bottom can be taken further. One can do this sort of lacing to generate in three dimensions, prisms, antiprisms, pyramids, and cupolae. The notion is that the surtopes of one face must systematically descend into surtopes of the other.

One can use two Wythoff mirror-edge figures from the same symmetry as the bases. When this is done, the side faces potentially appear at each of the nodes, being the lace-prism formed by all of the remaining nodes. Lace-prisms are useful, since the vertex figure of any Wythoff-mirror-edge figure is a lace prism with as many bases as the figure has maked nodes.

The symbol for a Wythoff lacing-prism is to write the top and bottom in sequence, and apply the &#x sequence at the end. So a dodecahedron truncated-dodecahedron lacing-prism combines x5o3o with x5x3o, as xx5xo3oo&#x. An antiprism is simply the lace-prism of a figure and its dual: for example, the cuboctahedral antiprism has as the top, o3x4o and as a base, o3m4o. The lace-prism is oo3xm4oo&#x.

For convex figures, we can describe a lace-prism as the convex hull, when the two bases are placed in parallel planes, sharing a common centre-perpendicular.

The dual of a lace-prism is a lace-tegum. This figure has its own description outside of saying 'dual of'… One places the two bases, and constructs pyramids, so that the apex of one base is in the centre of the other. The lace-tegum is the common intersection. If the resulting pyramid is not solid, then it is made solid by extension in the perpendicular. For example, if a pyramid is only in the x-y plane, it is extended throughout the values for z by way of a Cartesian product.

The antitegum, the dual of the antiprism, has every surtope an antitegum. An example of an antitegum is the measure-polytope, where every surtope is a simplex antitegum: lines, squares, cubes, tesseracts. But this holds true for all antitegums. This is because each face of an antitegum is formed by the antitegum on the face and the dual of the face.

An interesting figure one can create is an antitegmal cluster. Take any polytope, for example a dodecahedron. Each of its faces forms a pyramid radiating from the centre of the figure. We use each of these as one of the two lacing-pyramids. The second lacing pyramid is formed over the surface of the figure. This replaces each face by its antitegum. The inwards-pointing faces are not seen, and all that is seen is a apex of antitegums forming the second lacing-pyramid. The axies of the exposed faces connect the vertices of the dodecahedron with the vertices of the icosahedron. The antitegmal cluster of a figure is the same as that of the dual, and the whole surface is bounded by antitegums.

The most interesting of the antitegmal clusters is the one formed on the simplex. Complete with the innards, it is what happens when a measure-polytope is squashed so that the long axis is zero. The shape tiles space with relatively high efficiency, the dual tiling being one of a 60-degree rhombic tiling with additional planes perpendicular to the long axis.

An example of this is the digonal antitegum xo2ox&#m. Suppose the axis runs in the z direction. We construct a line pyramid (or triangle) in the x-z plane, and a second, inverted line-pyramid in the y-z plane. Were these not completed, all we would see is the common intersection in the z-axis. So the line-pyramid in the x-z plane exists for all values of y, and the y-z plane exists for all values of x. The common intersection is the space held between two vees of planes, which form pairs of faces of the tetrahedron.

The dual of a lace-prism is the lace-tegum, in terms of the symbols, a matter of swapping x and m where they occur.

Lace-prisms and lace-tegums can have any number of bases. When one projects a lace prism perpendicular to all of its bases, the bases appear as the vertices of a simplex. The base and apex of a normal pyramid would project as the ends of the line representing the altitude. Lacing-edges would project as edges of the altitude simplex, one for each kind of lacing. An example of a three-based lace-prism is oxx&#x. This is a square pyramid. The three bases are the apex and the north-south edges of the base. The east-west edges, and the sloping edges are different sets of lacing.

Laceland can be used with tilings as well. Although prism-products in general can not be applied to bollotopes (hyperbolic type polytopes), we can still consider lacing layers, either between the same, or different bollotope surfaces. For example, one can make a layer of triangular prisms by xx3oo8oo&#x. One can fill laminatopes with laceland style fillings. Laminatopes are discussed further on.

The vertex figure of any Wythoff mirror-edge figure is a lace-prism. It is quite possible to discuss the vertex figure of x3o4o3x in terms of a lace pyramid. It would become xo4ox&#x. The unmarked nodes form the transverse or base symmetry. The marked nodes correspond to the apices of the altitude. Each apex is connected separely to the bases.

For example, the x3x3o3o3x3o is a six-dimensional figure, and has a five-dimensional vertex figure. The transverse symmetry is o3o&o. The altitude has three vertices, forming a triangle. Each vertex of the altitude connects to the nodes differently. The first has no connection, ie o3o&o. This makes a point. The second connects to form a triangle, x3o3&o. The third connects as o3x&x or triangle-prism. The resulting lace-prism then is oxo3oox&oox&#x. We see that this figure has a three-dimensional transverse, and a two-dimensional altitude, all together, five dimensions.

The dual of the vertex-figure is the face of the dual. We write straight away, the face of m3m3o3o3m3o as omo3oom&oom&#m. This three-based lace-tegum is constructed in the same way as the two-based versions above, but is the intersection of three pyramids.

1.7 Semiates

The idea of semiates derives from removal of alternate vertices of a figure. A tetrahedron is a half-cube, for example. Semiation can be applied to higher values than two. For example, the semiated pentagon-pentagram prism is a regular figure called a pentachoron. What happens is that one reduces the vertices of a pentagon-pentagram prism so that only one-fifth of the vertices remain.

Semiation becomes more complex when there are more axies to pick from. This happens for the first time in six dimensions, where we see the threads on step-prisms and mod-prisms separate.

The notion behind semiates is that one can number the vertices systematically. When one takes a product of two or three such numbered polytopes, one forms an array (p,q) or (p,q,r). The idea of semiation is that one removes all those p,q which do not agree to some further restriction. For example, one might only want p and q equal, or the sum to be a multiple of some value. If the vertices are kept, the result is a prism. If the face-planes are kept, the result is a tegum.

In four dimensions, one finds the polygon-polygon prism. The vertices of a polygon can be numbered from 1 to p. In a polygon-polygon prism, this gives a set of p^2 points, running from 1,1 to p,p. What would happen, if we make these keep in step? The result is a large polygon. Instead of keeping in step, we can rotate one twice or x times faster than the other.

When p is a sum of two squares, such as 5 or 13, interesting things happen. The 1,2 bipentagon step prism is nothing more than the pentachoron. The 2,3 bi-{13} step tegum becomes a rather interesting polychoron, bounded by 13 identical sides. The matching step-prism has at least four vertices equidistant from the central one.

In six dimensions, the triple-product of polygons can be reduced in different ways. A step polygon makes everything step together, giving a polypeton with p vertices. The 1-2-4 tri-heptagon step-prism is the simplex in seven dimensions.

If one makes one dependent on the other two, for example x+y+z=0 mod p, then one has a polypeton with p^2 vertices. For example, the 1-2-4 tri-heptagon mod-prism contains the vertices of seven separate simplexes, for a total of 49 vertices.

Mod-prisms and mod-tegums get used in tilings as well. The body-centred-cubic can be viewed as a step-prism over modulo 2. In higher dimensions, one can use 3 or 4 in this place. The famous gosset-lattice in six dimensions can be constructed as step-prism over modulo 3, of a tri-hexagonal lattice, where the three numbered points are the vertices, the centres of the up-pointing triangles, and the centres of the down-pointing triangles. The corresponding mod-prism would place additional vertices in the centres of the cells of the {3,3,3,3,B,3}. This would make the vertices of o3o3m3o3oBo3o, a tiling of tri-triangular tegums, 720 to a vertex.

1.7 Laminatopes

A laminatope is a polytope bounded by unbounded faces. An example of a laminatope is a layer. The main use for laminatopes is to fill them with cells, and treated as a module for finding tilings. In Euclidean space, the layers are usually lace-prisms of tilings.

For example, xx3oo3oo3z&#x is a layer of triangular prisms. The xo3ox3oo3z&#x is an oct-tet layer. The etchings on both sides of this are x3o3o3z, a tiling of triangles. One can then stack these in all sorts of systematic orders to produce several different uniform tilings. For example, the oct-tet layers could advance, so that xo3ox3oo3z&#x is stacked on top of oo3xo3ox3z&#x. This advances the layer one step, producing a repeat after three layers. Alternately, one could treat the top surface as a mirror, and have layers of xo3ox3oo3z&#x and ox3xo3oo3z&#x. This tiling gives the hexagonal close pack.

The great search for uniform tilings centre on finding and sifting through the assorted laminate tilings. Many of the non-Wythoffian hyperbolic tilings are laminate as well.

1.8 The known uniform hyperbolic tilings

There are an infinite number of uniform bollohedra. John Conway and Chiam Goodman-Strauss have some generalised process for locating these. Without their notation, the process is a relative nightmare, since any given polygon in a vertex-figure can be replaced by a laminagon, and any two tilings can be merged.

Of the bollochora and higher, the picture is relatively simpler, although by no means complete. There are fourteen finite-extent groups in three and four dimensional tilings. These and a few star-groups, give rise by Wythoff mirror-edge construction to many of the known tilings.

There is an infinite family of borromeachora. For every polygon, except the square, one can create the matching borromeachoron. The heptagonal version has a dozen heptagonal prisms and eight cubes at each vertex. The vertex figure is an icosahedron, where the six edges parallel to the axial systems represent a {p}, and the remaining 24 edges that form the eight triangles are squares. We see in the case of the square borromeachoron, the result is the {4,3,5}.

There is also a scattered list of others.

One example is a partial truncation of the {3,5,3}. If selected vertices and attached edges are removed, these vertices become dodecahedra, and the icosahedra become pentagonal antiprisms. The vertex figure becomes a tetrahedrally truncated dodecahedron, with four dodecahedra and twelve pentagonal antiprisms at a vertex.

A second example is the laminatruncated {4,3,8}. The normal truncate produces an x4x3o8o, which has cells x4x3o 'truncated cube', and x3o8o. The x3o8o is not only infinite, but in this case, a planohedron, when the edges are equal. The surface can then be used as a mirror, to fill the whole of space with truncated cubes, 16 at a vertex. The vertex figure is an octagonal tegum, formed by rotating an octahedron by 45° around an axis.

The other known example is a development on o8o4xAx. In its primitive state, it has three kinds of cell: a planotope o8o4x, a curved o8o3x, and a rhombocuboctahedron x4o3x. The vertex figure is an octagonal rostrum, a prism with trapezoid sides. The o8o3x is the right size and curvature to be part of an oo8oo3xx&#x, an equilateral prismatic layer. This replaces the smaller octagon of the vertex figure with a cap of eight triangles. The base is completely flat, and can be used as a mirror. The resulting tiling has 16 triangular prisms, and 16 rhombocuboctahedra at the vertex. The vertex figure looks like a globe, with octagons forming the equator, the lines of longitude at 45° steps, and a smaller octagon representing 45°N and 45°S. Without the two poles, the thing can be made by rotating a cuboctahedron through 45° around the axis through the square-centres. One finds {4,8} formed by the squares passing through the great circles, and {8,6} formed by the octagons that can be drawn inside the rhombocuboctahedron, on the girthing hexagons of the two inscribed cuboctahedra.

Of four-dimensional tilings, two are known, these are duals of each other.

The first consists of a tiling of bi-truncated 24-chora o3x4x3o, 64 to a vertex. The thing derives from o3x4x3o8o, where there are two infinite cells x4x3o8o, and eight o3x4x3o at a vertex. The meeting-angle is smooth, and can be used to reflect the 45° angle occupied by the o3x4x3o around. This fills all-space. The resulting vertex figure is an octagon-octagon tegum, where 16 different x4x3o8o can be formed by one octagon, and an edge of the other. The cell walls are truncated cubes x4x3o, which form a laminatruncated {4,3,8}. The octagons form an {8,4}, and the triangles a {3,8}.

The dual is a tiling of bi-octagonal prisms, o8x2x8o, with 288 to a vertex. The vertex figure is o3m4m3o, formed by placing equal-sized dual 24-chora together, and covering the lot with the convex hull: 288 disphenoid tetrahedra. The squares form a tiling of {4,8}, and the octagons form an {8,6}, but there is no through-passing of three-dimensional cells.

2. THE POLYGLOSS

The Polygloss is a dictionary designed to encompass all of these concepts and more. Versions of it are placed on the web from time to time. One of the problems for it is that I have more words to describe than I have names for. There are many unnamed concepts that scream out for one.

Many have interim names. What I describe here as lace-prisms is in the Polygloss as exotic prisms. Exotic is used elsewhere. An exotic polygon has coincident vertices. The more useful concepts get interim names until they get their final name. Many of the others go by the hand-waving names, like 'thingie'.

For many years, the tegum product was called the octahedral product. The name does not fit well, but it was important even for hand-waving, that the thing had its own name. With tegum fully placed as the dual of prism, it provides a much richer and distinct name for many other figures.

2.1 The present terminology

The present terminology reflects the origins of geometry in the real world. It also carries useful concepts for which I am presently attempting to replicate in the Polygloss style. But it is in the main, a lost cause, I should imagine.

The same terminology in two dimensions carries across without modification to three. This seems to be the basis of some of the alternate vocabularies. A face, for example is a two-dimensional element in this style. Other things, like cells bound polychora.

While this provides a seamless conversion between dimensions, what gets lost is the auxiliary meanings. Apart from being a two-dimensional thing, planes divide. The present terminology is skewed in favour of the uniting forms.

Worse still, is the same stem gets divided into diverse meanings. A face and a facet in three dimensions, has the same meaning. In four, a facet might have several faces. In the Polygloss, a surface-mounted polygon is a *surhedron*, always. It *can* act as a face or a margin, but it always is a surhedron. Face and facet are then counted amongst the division-terms: a division between inside and outside.

The primary distinction in the Polygloss is to preserve the uniting or dividing nature of words, not the dimensionality. The whole thing is done in dual. A 6-edge under the dual becomes a 6-margin.

Whatever the virtues of the present notation is, it becomes a confusing and twisted maze when one tries to extend it to higher dimensions. For this reason, it was thought better to start afresh with terminology suited for a much higher dimension, and descend downwards. This is the view from six dimensions.

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