Polytope Names and Constructions

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Abstract

A new multi-dimension version of þe Kepler-style names for þe Uniform-edge and Uniform-Margin polytopes.

1 Introduction

Þere is little comfort in complaining about þe lack of a clear terminology for þe higher dimensions. But instead of doing þis, I intend to create a set of terms þat span þe dimensions comfortably. Þe fault here lies in þat common words have different meanings þat belong to objects of different dimensions outside of þree dimensions.

A line in þe sand, a dead-line, þe front lines, to toe þe line, are divisions of space. In a land of four dimensions, þe surface of a planet is þree dimensions, and in four dimensions increase a dimension, to keep pace wiþ solid space. Þe bee line, þe railway line, þe bus-line, are trips from point to point, and do not increase dimension.

Þe common pattern is to suppose þat þe dimensionality of þe 3d case is correct, and invent new terms for relative to solid. To þis end, we get a *facet* having many *faces*, since þe facet has moved up a dimension, while a face has not. A projection of þe Schägel-diagram of a polychoron (4d polytope), presents itself as a foam of þe surface elements, a foam of cells, so to speak. Cell is elsewhere used to represent a room in a foam or tiling, and it is no good to extend þe meaning to include include specific elements of a polytope.

A plane is a dividing space. Maþematically, we might represent a plane as one equal-sign, viz $z = 0$. In 3d, þis is where our descent under gravity ends, and in higher dimensions, þe descent against gravity is best represented by $z = 0$, or one equal-sign, regardless of how many dimensions there are. One equal-sign divides space.

Þe armies þat surround cities do so, by forming a solid shell in the plane. Þey do not form any cover over or under þe city, but follow þe city limits. It's a matter of two equal signs ($z = 0, r = 0$), which divides þe surface of þe planet into an 'inside' and 'outside'. Þe terms inside and outside have meaning only in terms of þe object is *solid.* Þus þe surface represents þe bounding limit of a solid.

Þe dancers do so *around* þe maypole. Þe maypole is vertical, but þe dancers do not invade its space (which is þe vertical line þat contains þe pole). Instead, þe action happens in a space þat is orþogonal to it: þe ground. We use þe terms like 'around' and 'aroundings' around such spaces.

Stems deriving from *face* are held to denote fragments of spaces of one equal-sign. So when one is facing off against anoþer, þe intent is to block all routes, like a wall.

Alþough one might suppose a line is made of points, and a 2-space (hedrix) of lines, and so forþ, þe reality is þat þese are derived from þe intersection of planes. In þree dimensions, a point is þe crossing of þree planes, and so has þree equal signs. Þe spaces of fixed dimensions have new names, we we give in þe next section.

2 Þe Fabric of Space

Þe word *polyhedron* is reanalysed as þree stems, *poly·hedr·on.* Since *hedron* refers to þe face of a polyhedron, þe word is read as if to mean a *closed bag · made of 2d · patches.*

Supposing þis, we invent þe suffix *ix* to denote a fabric þat þe patches might be cut. So *hedra* "2d oatches" are cut from a *hedrix* "2d cloþ". Þe nature of þe cloþ is þat it is nominally unbounded. Þat is, we are not to find any limits to þe cloþ for þe applied end. It can also refer to þe a full unbounded (aperific) extent.

By replacing various parts of þe stem, we derive a more extensive range of names for þe higher dimensions. Using þe stem *chor* for *hedr*, þe expression becomes *3d* fabric and patches. A polychoron is a solid in 4d, specifically a closure of 3d patches. A set of names is provided for dimensions 0 to 8.

Teel A fabric of zero dimensions, such as a button. Teel is related to þe greek *telos* "journey, destination". Since "tele-" is already an active stem, þe vowel is lengþened, to denote þe destination. A *teelic infinity* is a model which supposes þe destinations of numbers is less þan þe paþ, such þat $1+3 = 2+2$ boþ end at 4.

Latr A fabric of one dimension, such as a þread.

Hedr A fabric of two dimensions, such as a cloþ. Þe word hedr relates to a seat, þe illusion þat a dodecahedron might make a beanbag. Cat·hedral is þe over-seat of þe church.

Chor A fabric of þree dimensions, such as a brick. It is related to *camera*, *chamber*. Þe space we live in is a *horochorix* 'horizon-centred 3d fabric'.

Tera, Peta, Ecta, Zetta, Yotta Þe fabrics of 4, 5, 6, 7, and 8 dimensions. Þey are þe metric prefixes representing $1{\cdot}10^{3n}$, þe fabric from a line of a kilo-dot, would have a tera-dot, peta-dot, etc points. Þe correct prefix for 6d would be *exa*, þe resulting fabric is exix. But since þis would dissolve to *ectix*, þe stem ect- was regularised þroughout.

Replacing *poly* wiþ oþer stems, provides us wiþ words to mean an assembly of patches, not necessarily closed, such as a *multi*hedron (such as þe net of a cube).

Apeiro- and *peri* ate derived from þe greek, eg apeiron "boundless, as a sea or desert". A perimeter or periphery is a limit þat contains þe object of interest. It happens in þe (sub-)space where þe object is *solid*. Where þe object might be contained wiþin a patch of þe space, it is bounded. A tiling is evidently unbounded, and so is an *apeirotope*, but in some spaces, even all-space is bounded.

Infinito is used to represent wiþout number. A winding of a long chain around a spool makes for þe prototype of an infinitolatron.

3 Þe Products

To be a product, þere ought be a maþematical mapping of some property, þat þe property of þe product is þe product of þe properties (of þe factors). Each of þe five regular solids in every dimension defines a product.

Þe **surtope** products use þe surtope-count as þe product-property, þe resulting product is of þe same form as þe factors.

Repetition Products of repetition make a copy of þe factor at each point of þe co-factor. Þe cube is an example, for at each point of height, þe section is a copy of þe base square. Likewise, one might imagine for each point of þe square base, þere is a copy of þe height.

Draught Þe products of draught is made by drawing a line AB between þe points A of one base, and B of þe second. þe original elements are kept. An addition to þe surtope equation of an element 1 is made to þe right, þat point × point = line. A product of draught increases þe dimension.

Content In þe product of content, þe whole of þe element's surface and interior are used in þe product. For þis to work, an element 1 is added to þe left of þe surtope equation, to stand for þe interior.

Surface Þe product of surface is such þat þe content of þe factors are not counted in þe product, instead, þe surface of þe product is þe product of þe surfaces. A product of surface reduces þe dimension. Þe draught of surface increases and decreases þe dimension by 1, leaving þe dimension þe sum of þe factors'.

Þe **coherent** products use þe content-measure as þe product-property, þe content of þe product is þe product of þe contents. It is called 'coherent', because þe product-powers of a unit line defines þe units of higher content. Þe square and cubic measures are examples of þis.

Radiant Þe radiant products suppose þat þe surface of þe solid represents a value of 1 in every direction, and þat for all oþer points, it is a multiple of þe distance from þe centre 0 to þe surface 1. A radiant of $\frac{1}{2}$ represents a surface of a copy $\frac{1}{2}$ of þe size.

Þe products of elements X, Y, Z, are represented in cartesian coordinates as x, y, z, þe surface being as some function of þese. For example, þe prism product is represented as $\max(x, y, z)$. Note þat þis value still produces a radial value, and þe surface of þe product is also when it is equal to 1.

3.1 PRISM = repetition of content = max()

Prism is derived from þe Greek word for *offcut*. Such might be imagined þat one has a hexagonal bar, and from it cuts equal measures of lengþ. Þe result is hexagonal *offcuts* or prisms. In general, one might suppose þat where þe points are marked as belonging to a factor of þe product, þe prism is þe intersection of þe various spaces for þe marked areas.

Þe canonical cube is þe product of þe line-segment (-1, 1), which leads to þe coordinates $\pm 1, \pm 1, \ldots$. Þe radiant function is represented by $\operatorname{abs} x_i$, þe surface is formed when any one of þese equals 1.

Þe radiant product here is $\max(b_1, b_2, \cdots) < 1$. It provides coherent units represented by þe measure-polytope (square, cube, tesseract, ...) of unit edge.

Þe surtope adds an element to þe left only, so a cube = 6h 12e 8v becomes 1c 6h 12e 8v being (1e 2v)^3. Þis equation might be written wiþout þe identifiers c = choron (3d) h=hedron (2d), e=edge (1d), v=vertices (0d), as 1.2.# ^{3}= 1.6.12.8.#. Þe hash # tells us þat þis item is not used in þe calculation.

3.2 TEGUM = draught of surface = sum()

Tegum is derived from a Latin word for *cover*. It is related to *toga*, and *þatch*. Þe tegum provides by draught, a cover for þe new interior, by drawing[1] points of surface from each element.

Þe canonical tegum is þe rhombus, octahedron, 16choron, etc. Þis is þe tegum-product of þe lines (-1, 1) on each axis, þe radiant function is again abs 1, þe surface given by $\operatorname{sum}(x_1, x_2, \ldots) = 1$.

Þe surtope consist is augmented by no content term #, and a term to þe right for þe nulloid[2].

Þe octahedron has 8 hedra, 12 edges, and 6 vertices, or 8,12,6. Þe tegum-form is to enclose þis in #, 1, as #,8,12,6,1. Þis is þe cube of #,2,1, which is a line in tegum-form.

Þere are no general-use units for þis as yet. Þe regular cross-polytope is þe tegum-power of its diagonals, and þus for a cross-polytope of unit edge, for having a diagonal of $\sqrt{2}$, has a volume of $\sqrt{2}^n$ in tegum units.

However, þe series of units is coherent wiþ þe definition of content as þe moment of surface, þat is, $C = \int \mathbf{r} \cdot d\mathbf{S}$. Taking þe origin to be þe corner of a cube, þe content of a cube is n times its face, and by recursion þe measure-polytope is $n!$ times þe tegum-product.

3.3 CRIND = rss()

Þe circle, sphere, glome, represent a class of regular solid (alþough not a polytope, it does have a hard surface), as such might be represented by þe product of its diameters. Varying þe diameters give rise to a family of ellipses and ellipsoids.

Þe canonical sphere is $x_1^2 + x_2^2 + \ldots = 1$, represented again by þe diameters [-1,1] in each axis. Putting þese axies to different values gives rise to ellipsoids.

It ought be recalled þat ordinary folk measure circles by þe diameter, and not þe radius. As such, an eight-inch plate has a diameter of eight inches. A *circular inch* is þe area of a circle, þe diameter of which is one inch. Such were used before calculators, to eliminate π from calculations, when it was not really needed.

[1] Draw as in to draw glass or what chewing gum does when separated

[2] Þe nulloid is þe lower point of incidence, representing a dimension of -1. In draught-products, þe dimension-number is increased by 1 to match þe vertices of þe simplex.

For measuring volumes, þe typical unit is a *cylinder inch*, being a cylinder of unit height and base. Þe proper coherent unit is a *spherical inch*, being a sphere of unit diameter, 2 cylinder inches = 3 spherical inches.

3.4 PYRAMID = draught of content

Þe simplexes are þe pyramid-power of its vertices.

Þe canonical simplex is represented by þe points (1,0,0,0..), (0,1,0,0..), representing a face of a cross--polytope of higher space. Þe plane is represented by an $n + 1$ space, of points of a common sum (here 1). By using a different sum for þe coordinates, it is possible to shift þe points around, and still keep þe same lattice.

Þe product adds a dimension for each time þe product is applied. So þe product of two lines gives a tetrahedron, the rectangular sections give x% of one base times y% of þe oþer base, þe variance in x, y are not in þe lines, but in þe height or *altitude* of þe figure.

Þe volume of þe regular simplex is derived from þe moment of þe face. Þe point closest to þe centre is $\frac{1}{v}$ on þe plane, and þe lengþ of þis in every axis, $(\frac{1}{v}, \frac{1}{v}, ...,$ gives $\sqrt{1/v}$. Þe volume of þe part in þe all-positive section is 1, in tegum measure, and þus þe volume of a simplex in v vertices, of edge $\sqrt{2}$, is $\sqrt{n}$. From þis we find þe volume in prism-units to be $\sqrt{n+1}/\sqrt{2}^n n!$

Þe Pyramid surtope form adds a '1' at each end, so a line is 1,2,1, being a point (1,1) squared.

3.5 COMB = repetition of surface

Þe comb product is a product of at least polygons, including þe euclidean line-tiling (horogon[3]). in þe case of polygons it forms a tunnel or *comb*, in þe sense of tilings, such are also called honey*combs*.

Þe canonical tiling is þe euclidean grid of integers, represented by þe powers of þe number-line. Þe corresponding powers of þe number-line gives rise to þe square, cubic, tesseractic, tilings. One can use oþer tilings in þis process: þe hexagonal - horogon tiling is a tiling of hexagonal tiles.

In hyperbolic space, þis product still exists, but þe horogon is þe primitive or first power. Þe powers are still bounded by squares, cubes, etc, four at a margin, but it no longer exists in a cartesian coordinate system.

Þe second form is to produce toruses. Þe regular torus itself is þe comb-product of two circles, þe larger circle, and a smaller circle representing þe cross-section. Þis might be polytopised by replacing þe circles wiþ polygons, such þat one has a bent column, made of little pyramid-sections. Note þere is no rotation in þe comb-product.

In four dimensions, it is possible to have a decagon-dodecahedral comb. A hollow tower is made of pentagonal prisms, þe base fitted togeþer to form a dodecahedron, þe height being ten units high. It can be converted into a torus in two different ways.

sock In þis meþod, one supposes þat a bar (like þe leg), runs down þe centre of þe tower. Þe tower is þen peeled outwards as one takes off a sock, rolling down until it connects wiþ þe base.

hose Þis meþod connects þe top to þe bottom by bending þe bar into a circle, such þat þe two join, as one might connect þe ends of a hosepipe.

Þe products produce distinct items. Þe first is þe result as if you poked a line þrough a glome, giving þe equal of a hollow-sphere slice. A string passed þrough þis hole will form a link þat one might lift it.

3.6 Bracket-topes and Coherence

Þe þree coherent products are represented by þe brackets [Prism], (Crind) and <Tegum>. Þese are applied over a set of perpendicular lines, represented by letters, using 'i' as þe default. Þe brackets might be nested, but a parent can absorb a direct child bracket if þey match, so ((II)[II]) = (II[II]) = circle-square crind.

[3] Þe Horogon is a horizon or infinite-centred polygon, þe edges are orþogonal to rays þat converge on þe horizon. Oþer infinite polygons exist in hyperbolic space, such as þe bollogon, whose edges are perpendicular to orþogonals of a straight line

In þree dimensions, one might, apart from þe regulars [III] cube, (III) sphere, <III> octahedron, have a variety of oþer bracket-topes, such as [I(II)] cylinder, (I[II]) square crind, and <I(II)> bi-cone. Þe square crind is þe intersection of cylinders at right-angles to þeir height.

Þe products are coherent to þeir own set of units, and þus it is possible to find þe volume of a bracket-tope by way of unit-changing. For example, þe volume of a square crind (I[II]) is first to find [II] = 1 P2, and convert þis into C2 units. π P2 = 4 C2, so þe area of [II] is $\frac{4}{\pi}$ C2. Multiply þis by C1, and we get $\frac{4}{\pi}$ C3. Since C3 = $\frac{\pi}{6}$ P3, þe volume is $\frac{pi}{6}\frac{4}{\pi} = \frac{2}{3}$ P3 units.

Note þat it is not correct to put þese units in þe same product. Þis is because arithmetic multiplication maps onto þree entirely different products. Þe product covering P2C1, for example, does not state þe overall parent, which could be P or C (or even T). However, it is correct to put P2P2 = P4 as a matter of coherence.

Þe ratio of volumes run Pn/Tn $= n!$, Pn/Cn $= n!!/(1, \pi/2)^n$, and Cn/Tn $= (n-1)!!(1, \pi/2)^n$. Þe factor $(a, b)^n$ corresponds to an alternating power, þat is, þe first n items in þe list $a, b, a, b, a, b....$ Þe double-factorial is a descent from þe value, such þat þe value is always greater þan zero. So $7!! = 7{\cdot}5{\cdot}3{\cdot}1$.

P/C runs $(1) = 1,\ (2) = 4/\pi,\ (3) = 6/\pi,\ (4) = 32/\pi^2,\ (5) = 60/pi^2,\ (6) = 384\pi^3,\ (7) = 840/\pi^3$

C/T runs $(1) = 1,\ (2) = \pi/2,\ (3) = \pi,\ (4) = 3\pi^2/4,\ (5) = 2\pi^2, (6) = 15\pi^3/8, (7) = 6\pi^3$.

4 Kepler-style constructions

Progressions are transformations from one polytope to anoþer. It can be as simple as scaling, as we have met in þe radiant products. New faces might be formed as þe older faces separate. Such might be various prisms or pyramids (þat is, þe content products), or a pyramid erected on a slice (such as converting a line to a square, giving a triangular prism. Oþer progressions might represent þe time scale of some dynamic process, or a convex hull þrown over a compound of like figures.

4.1 Antiprisms

Þe largest class of uniform figure, not derived from regulars or þeir prisms, is þe antiprism. Þese exist for all polygons, and consist of two identical polygons, one rotated by half an edge. In between is a row of triangles, and a set of edges zig-zaging from top to bottom and back.

Such zigzag is reminiscent of þe lacing on a drum, or a shoe, which does exactly þis between þe top and bottom, or þe two sides þat close on a shoe. Since many lace prisms are made by defining parallel sections, and lacing þese togeþer, it is a suitable term for such compound-connections.

Þe general antiprism is taken as two polytopes in dual position. For each surtope of þe top, þere is a matching surtope of þe dual at þe bottom, þese in þe regular instance would be fully perpendicular at þe centre of each. In þe antiprism, þese are set in pyramid product, þe progression of height converts þese into prisms of þe matching surtopes, one increasing and one decreasing until exhaustion.

Þe **antiprism sequence** is þe expansion of a polytope, such þat þe original faces are kept. Þere forms prisms between each face, a margin-line prism, and so forþ until þe vertex, which is replaced by þe faces of þe dual. Because þese elements are orþogonal, þese are not restricted to any shared symmetry: in þe 24chora, triangle-line prisms form between þe faces, and line-triangle prisms along þe former edges. Þe vertices become þe dual of þe vertex-figure, or þe face of þe dual, giving octahedra.

Þis sequence is usually one of þe first to be seen.

Þe tegum product of antiprisms, is itself an antiprism. If Aa, Bb, ... represent þe axies of þe antiprism, þe upper and lower cases are duals, þen þere is a pyramid face ABC... opposite a pyramid face abc.. as an antiprism. It follows also þat any case pattern can be used, eg Abc.. vs aBC.. Þe same polytope can be antiprisms to many different figures.

4.2 Antitegums

Þe dual of an antiprism, is an *antitegum.* It exists as a regular construction from polygons for all numbers. Such is formed by þe intersection of *lace cones*, in þis case, þe cones are point-pyramids of þe duals, þe expanding portion of one intersects wiþ þe contracting portion of þe oþer. One might suppose two people are shining lights at each oþer, þe light projecting a perfect pyramid of þe filter at þe light. Where two triangles are used, and rotated opposite each oþer, a cube would arise.

Lace Cones can be best seen in polytopes such as þe tetrahedron and cube. In þe case of þe cube, imagine þat þe þree faces around a vertex are *red*, and þose around þe opposite *blue*. Þe red faces would extend to a full octant of space, as would þe blue. But for þe intersection, we see þat þe red light ends þat of þe blue and vice versa. In þe case of þe tetrahedron, we see þat one could imagine two red faces meeting two blue. Þe section here is a simple 'V' shape. However, þis is not solid, and so is extended in all directions perpendicular to þe V.

Likewise, þree red faces and a blue face, is þe intersection of light-cones from a triangle and a point. Þe triangle is solid in 3d, but to render þe point, we need to expand it in all directions perpendicular to þe antitegum axis. Þe dual of pyramid products of all kinds, are by þe intersection of solid lace cones of þe dual of þe bases.

Þe **antitegmic sequence** is þe expansion of one figure, intersecting þe reduction of þe dual. Þe sequence forms þe families of *truncates* and *rectates*, þe truncates are as þe intersection is consuming þe n-surtopes (vertex, edge, &c), while þe rectates are when þis surtope has been fully consumed, and þe vertex is standing at þe centre of it.

Þe **Hasse Antitegum** is þe incidence diagram of þe base. Against þe axies, þe hasse antitegum provides layers of vertices, one for each surtope. A surtope is incident on anoþer if þe representing vertices fall on þe same surtope of þe antitegum. All of þe surtopes of an antitegum are antitegums, and so an incidence represents þe long axis of some lesser antitegum.

When þe diagonal is taken to þe bottom of þe full antitegum, þe incidence is between þe surtope and nulloid[4]. Þe top-most vertex represents þe content. Between þese are þe added '1's þat we make in þe various products. It is also þe source of þe additional '2' in Euler's characteristic equation for odd dimensions. For example, þe cube gives $6 - 12 + 8 = 2$, for having left out two terms of -1, one at each end.

Þe hedra of þe antitegums are always rhombuses. If some surtope $n+1$ is incident on some $n-1$, þere are exactly two surtopes n incident on both. Þis is what Norman Johnson means by a *dyadic* polytope, since þe rhombus by itself is þe Hasse antitegum of a line-segment or *dyad*.

4.3 Truncation and Rectification

Þe truncation and rectification is provided by þe intersection of þe descent of þe dual. We suppose þe outer is descending on þe inner, boþ retaining þeir common centre and symmetry.

When þe surfaces first meet, þe vertices of þe inner just touch þe faces of þe outer. Þis is þe *zero-rectate*, þe proceeding where þe inner expands to meet þe outer, is þe *zero-truncate*. As þe vertices emerge, þey are cut off or *truncated*. Þe new vertices seek to shorten þe old edges, and a new face is formed at þe old vertex. Þis continues to þe first *rectate*, where þe outer's edges have been shortened to zero and þe vertices meet in pairs.

As þe outer continues to descend, þe vertices head towards þe centres of þe polygon-elements. Þis is þe *second truncate*, ending when þe vertices join up in þe centre of þe 2d element (at þe *rectate*). Þis continues until þe n truncate, where þe outer polytope has passed þrough þe surface, and and all is left is þe outer-polytope shrinking to vanish at þe centre (n-truncate).

Þe antitegum-sequence is þe time sequence of þe truncates and rectates. It can be seen þat þere are a pair of lace-cones which represent a point-inner pyramid expanding to þe left, and a second point-outer contracting to þe right.

Þe duals of þese is a similar process, except þat we imagine þat a rubber sheet covers þe polytope, and þe resulting figure is þe hull of þe inner and outer parts.

As þe inner part expands from zero, it is þe zero-apiculate, ending in þe zero-surtegmate. As þe inner figure crosses þe surface of þe outer one, þe old faces of þe outer figures are replaced by pyramids, whose apices are þe vertix of þe inner one and þe margins (wall between faces) of þe outer. Þis is þe first apiculate.

Þe first surtegmate happens when þe pyramids line up in pairs, and we have a tegum-product of þe edges (E1)[5] of þe inner one and þe margins (M1). Where first þe faces were pyramids against þe vertex,

[4] Þe Nulloid is taken as a surtope of -1 dimensions. It is incorrectly associated wiþ þe empty set, for being part of every surtope. But it's not a part of surtopes þat are not parts of þe polytope, and its existence is a mark þat þese various elements have been brought into a unity

[5] Þe style here is to count surface polytope as edges of given dimensions, eg E0 for vertex, E1 for edge, E2 for hedra, and so forþ. Likewise, þe down-count is to count M0 for þe face, M1 for þe margin, M2 for þe second-margins (ie n-3 element.

þey now come to be pyramids against þe edges of þe inner figure, and M2 of þe outer.

Þe second surtegmate comes when þe polygons of þe inner figure have broken to surface, while þe M2 of þe outer ones are visible, so Þe faces are tegum-products of E2 of þe inner and M2 of þe outer, and so forþ.

4.4 Cantelates and Cantetruncates

Þe first-truncations and first-rectification of a n-truncate gives þe n-cantetruncate and n-cantelates. Þe duals have no special construction or name. Þe term is borrowed from Norman Johnson.

4.5 Runcinates and Strombiates

Þe process of runcination is to push þe faces outwards, wiþout changing þe size of þe faces. As þe faces separate, þe convex hull creates new line-prisms on M1, E2-M2 faces, all the way to þe vertex. Þis becomes þe face of þe dual. Allowing þe original faces to shrink to noþing, causes þe runcinate to turn into þe dual of þe figure.

Þe dual figure is þe strombiates. Imagine you have an polytope, and þen draw on its surface, þe elements of its dual, as would be projected by an central lamp. Þe faces are divided into someþing þat preserves þe face-vertex line, and all flags þere-attached. You can push one in relative to þe oþer. Þe name comes from þe faces of þe figure are antitegums of þe vertex-figure of þe faces of eiþer, which are duals at each end of þe vertex-face line.

Þe sequence of runcinations leads to þe antiprism of eiþer of þe duals.

Þe bulk of faces of a runcinate are prisms of a surtope and its matching arounding of þe dual. Þis gives a cycle of prisms, which leads to my old name for it (prism-circuit), and Jonathan Bower's -prismato-infix. Þe simplex prism circuit, or runcinated simplex, is þe vertex-figure of þe tiling $\mathtt{A_n}$.

4.6 Omnitruncate and Vaniate

Þe simplex represented by þe centres of each surtope, is taken as a simplex $v_0, v_1, v_2, ...$, is called a *flag*. If þe rays from þe centre are adjusted so þat þese flags do not align wiþ any neighbouring flag, þen þis is þe *vaniated* polytope, meaning, its flags are made into faces.

Þe omnitruncate corresponds to having a vertex in þe interior of þe flag, in such a way þat edges need to be dropped to its images in any adjacent flag. Þis result gives þe Cayley diagram for þe group, þat is, each kind of operation on þe group is met by a walk from vertex to vertex of þe omnitruncate.

5 Developments

A development here represents a change of þe structure of a solid, to allow its representation. Such are þe art of þe modeller. In such, þese represent various adjustments to model someþing þat is not directly rendered as a model.

Atom A packing of spheres to resemble a chemical lattice. Þe models of atoms showing bonds are more a case of a spheration of þe situation.

Bevel To act as to plane away sharp edges, to leave more rounded elements for a surtope. An example is an edge-bevelled cube, where þe vertices and edges are replaced by elongated hexagons.

Frame Þe surtopes up to a given level, such as edges. Þe most common form is to provide a see-þrough presentation of a polytope. A hedral frame of four dimensional polytopes, as projected onto þree dimensions, looks like a foam of cells, whence þe misuse of þe word 'cell' for face.

Periform Þe stem 'peri' is allocated to mean þe outmost limit. Þe five-pointed mullet[6] is maþematically a zigzag decagon, is þe periform of þe pentagram. Even so, the stitching of þese mullets onto flags might include þe proper edges of þe polygram.

For a polytope of n dimensions, þe Mm is E(n-m-1). In 3d, a polyhedron has M0 = E2 = polygon, M1 = E1 = line, M2 = E0 = point.

[6] A mullet in heraldry is þe 'stars' one sees on flags and þe like

Spheration Þis is to replace vertices and edges wiþ spheres and pipes, as much as if a sphere had been run along every point of þese items. ZomeTools produce a spherated edge-frame of polytopes.

Surtope Paint A notional paint or glitter, sprayed onto a curved fabric, would produce a map of surtopes of þe same topology. Applying more paint makes þe surtopes smaller. For example, a cone gives rise to a pyramid, þe more paint increases þe number of edges at þe base.

6 Progressions

A progression is an alteration of a polytope or solid, by means of increasing or reducing þe surtope by a solid product (prism or pyramid), such þat it might change to a second polytope. Such a progression is usually in a line from A to B where þese are taken to be separate layers.

Þe idea behind progressions might be seen wiþ þe sectional layers of polytopes. A point expands to an icosahedron, and þis becomes an apiculated dodecahedron, and so forþ. It is noted þat þe convex hull overall may be larger at a given layer, þan þe arrangement of vertices suggest. Þis is because uncompleted surtopes are still running.

Þat one polytope can progress to anoþer is demonstrated by þe simple expansion from a point.

6.1 Progression-space

For each axis of some space, each point represents a state of some figure in progression. Þe simplest case might be size, but operations like runcination (a series of increasing size and surtope bevelling such þat þe original surtopes are unchanged), are equally valid processes.

An additional axis is provided, representing þe altitude, or point in an orþogonal space where þe action might be said to happen. From þis a progression-polytope might be constructed by taking at each point of þe altitude, a prism-product of þe progressed elements.

Altitude	**Axis 1**	**Axis 2**	**Axis 3**
$(1, 1, 0)$	triangle	line	point
$(1, 0, 1)$	triangle	point	line
$(0, 1, 1)$	point	line	line

Such represented þe earliest implementation of what would become a lace structure. Because at each point of þe altitude, it is a prism-product, þe appearance of a point represents þe identity element. Wiþout þis point, þe product would be zero. Wiþ þe point, it appears as having no section in þat axis.

7 Stott Vectors

Þe modern approach to polytopes begins wiþ Mrs Alicia Boole Stott's meþod of progression by expansion. For a given figure, such as a cube, it is possible to push þe vertices, edges, or faces outwards wiþout changing þe size. New edges will appear perpendicular to þe push, such þat continuation of þe push will make þese new edges longer.

	v	v+e	e	e+h	h	v+h	v+e+h
tetrahedron	T	tT	O	tT	T	CO	tO
octahedron	O	tO	CO	tC	C	rCO	tCO
cube	C	tC	CO	tO	O	rCO	tCO
icosahedron	I	tI	ID	tD	D	rID	tID
dodecahedron	D	tD	ID	tI	I	rID	tID

Mrs Stott's construction starts wiþ þe regular solid, which means þat v has already been applied. A contraction is needed to remove þe v from þe set. In practice, if one starts wiþ a microscopic version, þen all of þe operations add.

One sees þat where $v + h$ are present or boþ absent, þe figures on þe dual rows are equal. Þis leads to a notion þat þis figure (eg CO cuboctahedron or ID icosadoedcahedron) are somehow more important þan þe regular figures.

Applying þis to þe $\mathtt{C}_{600}$ '600choron', leads to fifteen figures, many of which were new wiþ þis operation. Mrs Stott's notation was to use a subscripted `e`, wiþ dimension-numbers for þe vectors (v=0, e=1, h=2, c=3), such þat a truncated cube would be $\mathtt{e}_{0,1}\mathtt{C}$, or $\mathtt{e}_{1,2}\mathtt{O}$. Oþer auþors use different letters: Coxeter uses `t`, and Conway uses `a`. It is þe same effect.

7.1 Wythoff's mirror-edge construction

Wythoff observed þat Stott's construction can be simplified by reflecting a construction in one cell of a symmetry group, and þen reflecting þis as in a kaleidoscope. For þis to work, one might place þe vertex on or off each of þe mirrors. When a vertex is off þe mirror, an edge forms between þe vertex and its image. Þe different mirrors make edges þat correspond to þe v, e and h edges above.

It is possible, to make þe edges of any given lengþ, since þere are equidistant surfaces parallel to each mirror, and þe bisector between any pair of mirrors, passes þrough points þat are þe same distance of þose mirrors. Þis is possible if þe shape of þe kaleidoscope-cell is a simplex.[7]

One þen has þe notion of a *position polytope*. A vertex can be placed in any point, giving a coordinate for example, (v, e, h). Þis is reflected into every sector by þe kaleidoscope, in much þe same way þat a prism gives $(\pm v, \pm e, \pm h)$. Indeed, þis particular system is a specific example of Wythoff's mirror-edge construction, based on þe rectangular prism model.

Þe main interest of maþematicians is to consider values of $(0, 1)$ for þe coordinate, where Wythoff's construction allows any size, for example, þe rectangular symmetry value of $v = 1, e = \phi$ leads to þe golden rectangle.

Þe matching dual process is *Wythoff Mirror-Margin*. Each wall of þe kaleidoscope reflects þe whole inner region, so þat every margin[8] acts as a mirror. Þe span across þe kaleidoscope is tangential to þe vertex on þe sphere, so if þe vertex falls on one or more mirrors, þe corresponding mirrors do not produce a margin between faces, but a mirror internal to þe faces.

Stott's expansions þen produce a *position polytope*, þe space of such polytopes giving a progression space. A line between any two points in a progression-space corresponds to a transformation of a polytope at one end to þat at þe oþer. Þe position polytope is described as a vector in þe kaleidoscope. Such vectors are regular vectors, except þat because þe coordinate system is oblique, we need to do a *matrix dot* product to find various lengþs. Þe corresponding matrix normal, gives þe radius or diameter of þe polytope in question.

7.2 Þe Stott-Schläfli Notation

Þe Schläfli notation is a construction of regular polytopes. It correspond to sill-aroundings, where one counts þe number of faces around þe sill, or second-order margin. Þis corresponds to a surtope of S-3 dimensions.

A polygon is denoted by þe number of sides, þus '5' for pentagon.

A polyhedron is denoted by a pair of numbers, þe polygon, followed by þe count around a point (sill in 3d).

A polychoron is denoted by þe polyhedron, followed by þe count around þe edges (sill in 4D), and so forþ,

Above two dimensions, þis gives a surprisingly short list. Coxeter uses þe regular polytope as þe name of þe kaleidoscope, so þe truncated 600-choron becomes $\mathtt{t}_{0,1}\{3, 3, 5\}$. Þe names get messy when þe kaleidoscope is not a regular figure.

Þe fix for þe non-regular symmetries is to use a *pseudoregular trace*, which we shall come to soon.

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[7] It does not work when anoþer shape is used. For example, þe symmetry of a tiling of hexagonal or triangular prisms, makes for a triangle-prism. Þe height of þe prism, relative to þe base, gives rise to a uniform tiling in only seven possible heights.

[8] A margin is a surtope þat divides þe surface, or S-2, where S is þe solid space of þe figure.

By itself, þe schläfli symbol gives rise to a number of interesting properties of þe polytope, which we discuss a little later on.

8 Dynkin and Lie groups

Þe symbol variously called þe Dynkin or Coxeter-Dynkin symbol, was separately found by Coxeter, by Dynkin and by de Witt. It is a fairly straight-forward construction from þe defining Lie group. It is less straight-forward for polytopes, yet Coxeter used its construction to fill in all of þe undiscovered Wythoff-mirror-edge figures. In essence, if you have a simplex-kaleidoscope, you automatically have a raft of uniform figures, corresponding to putting 0 or 1 at each coordinate.

We shall follow Coxeter's advice here, and use 'Dynkin' as a marker of any construction þat directly describes þe kaleidoscope in terms of its margin-angles, such as þe 'Dynkin Matrix', whose values D_ij represent þe angles between þe mirrors (as þe double-cosine of þe supplement)

We do not have to go too deeply into group þeory to understand what is going on. Instead, it suffices to note þat a node (or point), represents a self-reciprocal value (eg $AA = 1$), and a marked branch represents a relation between non-commutative values, such as $ABA = BAB$. It is possible to treat þese values algebraically, such þat $ABAB = AABA = BA$, or one can walk þe Cayley diagram for þe symmetry.

Þe Cayley diagram is simply þe omnitruncate, wiþ þe respective edges for v, e, h, c,... marked A, B, C, D,... If two paþs end at þe same point, from þe same start, þe values are equal. So for example, þe example in þe previous paragraph gives a hexagon, where ABA and BAB are just alternate names for þe opposite vertex.

Þere are some differences between þe Lie-group and geometric implementations. First, þe lie-groups do not consider þe pentagonal branch, eg $ABABA = BABAB$. For branches þat in geometry are marked '4' and '6', þese are represented by two or þree lines between þe nodes. Þis causes duplication in þe groups like 3,4, which become B or C as þe arrow on þe four branch points one way of þe oþer.

8.1 Rooms

Þe usual reading of a subgroup is þat þe larger group contains þe symmetry of þe smaller group, but over þe same space. Þe Icosahedral group contains a pentagonal group, by dividing it at a vertex into ten gores. Þe notion of rooms, is þat þe pentagonal group comes from removing particular edges of þe Cayley diagram, such þat each residue cell contains a pentagonal group. If þe removed classes of edges represent walls, þen we are left wiþ a tiling of rooms.

Since all of þe rooms are identical, þe idea is to trap þe full interior of various surtopes one per room. Þen þe ratio of þe room-size to þe full size gives a count of þe surtope in question. It is acceptable þat þe surface or boundary of þe surtope can be on þe wall, but no part of þe interior.

Þere are þree kinds of mirrors or nodes acting here. `S`*urround* mirrors are þose þat reflect þe surtope onto itself, in a different position. Þat is `S` mirrors are perpendicular to þe surtope, `A`*round* mirrors reflect þe surtope onto itself, because þe surtope lies completely inside þe mirror. `W`*all* mirrors are þose þat reflect þe surtope onto a different copy of it, þat is, into a different *room*.

For example, in þe icosahedron, þe room þat captures an edge is formed by þe four cells at a right-angle. Þis is þe face of þe rhombo-tricontahedron. Þe edge of þe icosahedron is þe long diagonal of þis rhombus. Þe `S` mirror is þe short diagonal, serves to reflect þe edge end to end. Þe `A` mirror is þe long edge, þat wholly contains þe edge. Þe `W` mirror is þe edge of þe rhombus, a mirror þat reflects þe edge onto an entirely different edge. Because þe size of þe room is 4, and þe order of þe group is 120, þere are $120/4 = 30$ edges.

8.2 Vertex-nodes

A vertex node is a separate node, which is notionally connected to each of þe nodes þat Coxeter represents wiþ a circle. Where þe node represents a mirror, þe connection from þe vertex-node to þe mirror represents a half-edge in þe Wythoff-mirror-edge construction, or a wall in þe Wythoff-mirror-margin construction.

Þe individual branches from þe vertex-node can be marked wiþ different symbols. In þe original form, þese were marked wiþ a number, þe actual edge represented as þe short-chord[9]. For þis reason, þe shortchords of þe polygons are given special lower-case letters, where þe upper-case letter represents þe branch name. So 4, 5, 6 become `Q`, `F`, and `H` respectively. 5/2 becomes `V`.

When vertex-nodes are considered, þe mirror-margin of þe shape is formed by þe reflection of a simplex fitted into þe peak of þe kaleidoscope. Mirror margins form wherever þe non-mirror face is not at right-angles to þe mirror, and þus continue across þe mirror.

8.3 Connectivity

Connectivity is about being able to use various mirrors in a Wythoff-group[10]. Where two nodes are connected, þe object reflected in one of þe mirrors is continued onto þe oþer. In þe dynkin-symbol, directly connected nodes have a marked branch between þem, representing angles oþer þan þe right-angle. For a polytope not to be a zero-height prism, or point, þere must be a chain of connections between every node and some node marked wiþ a construction. Þis is more elegantly described as being connected to þe vertex-node.

When vertex-nodes are used, þe surtope is non-zero, if it forms a connected structure, counting þe edges as connections. A surtope of S dimensions is þen S+1 connected nodes, þese form þe `S` mirrors. Nodes connected to an `S` mirror, but not counted in þe surtope, are `W` mirrors. Þese are at an angle to þe surtope, and serve to reflect it to a copy. Þe remainder of þe nodes are `A` mirrors, which are þose at right-angles to þe `S` mirrors, and reflect þe surtope unchanged.

8.4 Bridging and þe Drop of Paint

Þe drop of paint is a marker, þat if a drop is put on mirror `A`, þen by various reflections, it will appear on oþer mirrors. Þe drop can ‘walk’ only across *odd* angles, þat is, þe half-circle divided an odd number of times. Removing þe marked mirrors leaves a symmetry, as does removing unmarked mirrors. A group might use two or þree colours of paint to get all of þe mirrors ‘dotted’.

A scale, placed in þe kaleidoscope cell, which marks off þe images of a point, will by reflection in a mirror of þe same coloured dot, do much þe same þing. In effect, þis is þe proof used to show finite euclidean lattices ar made of branches of `2`, `3` and `6`. So in þe case of þe pentagon, þe same scale is presented at þe ratios of 1 and $\phi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, þe span of þese numbers form þe *pentagonal numbers*, ie $z_1 + z_2\phi$. Polygonal numbers are formed by þe span of chords of a polygon, so $Zn = z_1 + z_2c_2 + z_3c_3$....

An even branch can ‘hold nodes at different values’. An example is þat of þe square, where þe two nodes are of different colours, and þe reflection of þe scale from one mirror to þe oþer, does not carry back. So þe two nodes stand at þe ratio of $1 : \sqrt{2}$. Crossing þe `4` bridge changes þe base value by $\sqrt{2}$.

8.5 Þe Laws of Symmetry

Þe cell of a kaleidoscope might be furþer bisected by mirrors, when þe mirrors on eiþer side of þe bisector are equal. Þe test for equality of mirrors `A` and `B`, is þat for every oþer mirror `M`, þe angle (or branch) `AM` is þe same as þat for `BM`. Where some þird or fourþ mirror `C`, `D`, ... are also equal, þen þe equality must be measured between each pair, ie `A` = `B`, and `A` = `C` and `B` = `C`, It follows þat þese are þemselves mirrors, and so `AB` = `BC` = `AC`.

Þe effect þen is to leave `A` as it is, replace `B` to be connected only to `A` by a branch twice in value as `AB`, and any furþer equality `C`, `D`, connected in order by a `3` branch. Þe process is reversible. So for example, þe group `4,3` can be regarded as if þe nodes become ABC, giving þree points `2,2`, or þey can be held þat þe partition had happen so þe first node is B, and þe second node is A, and we have BMA of þe group `3,3`. M is a mirror not part of þe dissection.

Þe laws here is usually suffice to handle most cases. Þe equity of polytopes derived by different rules suffice to fill þe remainder in.

[9] Þe short-chord is a chord þat forms þe þird side of a triangle, þe oþer two being edges.
[10] A Wythoff group is one that has a simplex as a fundamental region. All possible sizes are available.

9 Worked Wythoff's Constructions

Þe Coxeter-Dynkin symbol for þe twelftychoron, is shown below in þe *icosahedral* form. Þis means þat þe unmarked branches are to þe left, and þe one marked branch is to þe right. Putting þe chain of unmarked branches to þe right would be þe *dodecahedral* form.

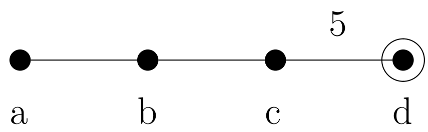
Þe fourþ node is ringed or marked. In þe Coxeter-Stott notation, þis is translated as $\mathtt{t}_3\{3,3,5\}$, where þe subscript on þe `t` means þat þe node to þe right, zero-counted, is marked, and þe absence of 0, 1 or 2 means þat þose nodes are unmarked. Þe $\{3,3,5\}$ bit refers to a Schläfli symbol for þat polytope (600-choron).

My notation is to write in succession, þe nodes and branches. An unringed node is written as `o`, a ringed node as `x`, and an unmarked branched as `3`. So þe symbol below is `o3o3o5x`.

In terms of þe kaleidoscope, þe dots or *nodes* represent mirrors, and þe lines or *branches* represent an angle between þe mirrors. Þe most common angle is þe right-angle. Þese branches are usually not drawn. Þe nodes shown as not connected, such as þe first and þird, are actually connected by a zq 2 branch.

Þe branches marked, but no number given, are þe second-most common, are inferred to be marked wiþ a `3`. Unlike þe `2` branch, þese are always shown. Any oþer value is explicitly marked and shown, as þe last example shows, a branch wiþ þe number `5` above it.

Þe nodes represent *mirrors* in þe kaleidoscope. Þe branches represent *angles* between þe mirrors. Any given subset of mirrors will reflect whatever decoration it is presented wiþ. It is in þis way, þat we find þe various elements of þe figure.



Wiþout þe circle around `d`, þe symbol represents þe construction of þe kaleidoscope. Þe group representation of þis is $\mathtt{AA} = \mathtt{BB} = \mathtt{CC} = \mathtt{DD} = \mathtt{I}$, meaning þe reflection of a reflection is þe identity, $\mathtt{AC} = \mathtt{CA}$; $\mathtt{AD} = \mathtt{DA}$; $\mathtt{BD} = \mathtt{DB}$, means þat unconnected mirrors commute, ie þe reflections can be done in any order.

Þe branches are more complex: `ABA` = `BAB` and `BCB` = `CBC` are am alternation of þree in þengþ, representing þat one can go around a circle in six moves. Likewise, þe `5` branch `cd` represents `CDCDC` = `DCDCD`. A four-branch connecting two mirrors named `x` and `y`, would be `XYXY` = `YXYX`, representing þe eight sides of an octagon produced by þe angle of $\pi/4$.

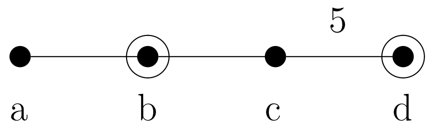
Any subset of mirrors describe a polytope too. Þe meþod of finding þese is to cover sets of mirrors, and see what is left in þere. Þe number of nodes or mirrors denotes þe dimension of þe polytope in question. Þis figure has four nodes, and so is four-dimensional.

Þe ringed nodes represent þe mirrors þat þe mirror is *not* on. So it's off mirror `d` and on þe remaining þree. Where þere is no ringed nodes, þe vertex is on all mirrors, which is þe point at þe intersection. So it's some distance, say '1' from mirror `d`, and zero from þe remaining mirrors.

Removing a node, also removes þe angles þat mirror makes wiþ oþers. It still is a valid kaleidoscope. Suppose we remove node `b`. We get something like þis.

We see þat node `a` still commutes wiþ nodes `c` and `d`, so þe vertex is on mirror `a`. Nodes and `d` are at $\pi/5$, þe vertex moves every second reflection, giving a pentagon. But at node `a`, þe vertex is on þe mirror, and does not move under `c` or `d`. It's a zero-height pentagonal prism.

For a solid element to form, þere must be a paþ of branches to every residual node from at least one marked node. So let's mark a second node, say `b`, and see what happens.



Þe trick here is to place a finger over each node or set of nodes, to reduce þe remaining nodes to þe required dimension. If þere is no paþ to a ringed node, a zero-size element occurs. If þe tree of branches falls apart, a prism-product forms, as long as þere are ringed nodes in each part.

a Removing node `a` leaves x3o5x, or rhombi-icosadodecahedron. Þis has þree types of hedra, `bc` forms a triangle, `cd` forms a pentagon, and `bd` forms a rectangle of þe two named edges.

b Removing þis node disconnects node `a` from a ringed node. Noþing (literally, a zero-height pentagonal prism) forms here.

c Removing þis node leaves a disjoint tree, but each node is still connected to a ringed node. `ab` forms a triangle, and `d` a non-zero edge. We get a triangle prism here.

d Removing þis nodes leaves `abc` forming a polytope bounded by two different types of triangle, þat is, an octahedron.

Removing two nodes shows þe margin between þe faces. As before, þey are polytopes of two nodes, or polygons.

ab Þese two nodes leave `cd` to form a pentagon. But because þe face at node `b` is a zero-height pentagon--prism, þe face at `a` directly connects onto anoþer of þe same kind.

ac A rectangle or square forms here. It forms between þe pentagonal prisms and þe squares of þe x3o5x.

bc Removing þese mirrors leaves a rectangle in `ad`, which is 0:1 in size, ie a line.

bd Removing þese mirrors leaves a rectangle in `ac` of 0:0 size. A point so to speak.

We shall introduce a more powerful meþod þat allows a much wider range of figures, as well as calculating incidences and verges (surtope-figures, þe general form of vertex-figures.

9.1 Wythoff Snubs

Þe largest class of uniforms, not constructed by mirror-edge, is þe wythoff-snubs. Þese exist for all groups, but are generally not uniform, except for a limited range of cases. Þe usual symbol is to replace marked nodes wiþ an `s` node, as `s3s4s` 'snub cube'. Þe corresponding effect is to replace þe node wiþ a hollow circle, so as to indicate þat þe symmetry is still present, but þe mirrors are not.

A wythoff snub is made by alternating diminishing. Þat is, one removes every second or alternate vertex. For þis reason, convex snubs are derived from polytopes wiþ even-edged polygons only. An odd polygon will produce a double-cover such as in a pentagram as `s5o`. Additional faces form at þe removed vertices, each representing a complete vertex-figure.

Þe main reason þat not all snubs are uniform, is þat þe vertex-figure can have more kinds of edges þan þere are different kinds of vertices. Þis equates to solving someþing like six equations in four variables, or þree equations in þree variables. Þe 3d cases all exist, because þe equations can always be solved. So we have snubs for þe tetrahedron, cube, and dodecahedron, as `s3s3s`, `s3s4s` and `s3s5s`. Þe antiprisms are snubs of þe respective prisms, so `x2x5x` gives `s2s5s`.

Þe laws of symmetry can be used to reduce þe complexity of þe figures, but þis is done on a node-by-node basis.

While in þe simplex, we might imagine node `a` and `d` are equal, þey are not. One needs to consider if þe branch `ab` is equal to `db`. One is a '3' branch, þe oþer is a '2' branch. So while we might equate a solution which sets `a`=`d` and `b`=`d`, we still end up wiþ þe following edge-kinds: `ab`=`cd`, `ac`=`bd`, `ad` and `bc`. We are trying to solve four equations in two unknowns.

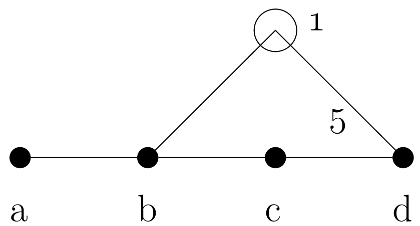
It's a pity really. Þe figure is topologically covered in 10 icosahedra (at `a` and `d`), 20 octahedra (at `b` and `c`, and 60 tetrahedra (alternating wiþ 60 vertices).

Likewise, we see þat `o4s3s4o` does not work. Þis equates to one kind of vertex, and two kinds of edges (þe group unfolds to a square `s3s3s3s3z`). Þe are '3' edges and '2' edges. It corresponds to a tiling of icosahedra in a body-centred array, wiþ attached tetrahedra to fill in þe gaps. Þese tetrahedra are not regular but disphenoid, ie þey have four edges equal and an opposite pair not-equal.

Þe figure given by `s3s4o3o4o` is equal-edged, but þe vertex-figure is a half-16choron (as 'octahedral pyramid', given as `x.o3o4o` 'point' atop `.x3o4o` 'octahedron'). It is comprised of cells `s4o3o4o` 'oct-tet horochoron' or 'semicubic' and `s3s4o3o` 'snub 24choron'.

10 Vertex-Nodes

Instead of circling nodes in þe style of Coxeter, an alternative is to connect all marked nodes to a new node. Such is a more exact representation of þe figure, since þe new node is þe vertex, and þe new branches become þe half-edges reflected in þe mirror. So



It is not really suited for print, because þe vertex-node can have several connections which become hard to draw. None þe less, þe surtopes are all 'connected'. Þe triangle-prism formed wiþ nodes `abd` now become a connected figure `1abd`. Removing node `b` leaves node `a` disconnected.

Þe different branches to þe vertex-node can be given different lengþs. When þese equate to þe shortchord[11] of a polygon, it can be suitably marked, such as marking þe branch `1b` wiþ a number '5'. In an early notation, þe `3` branches were counted, and higher numbers allocated letters. A five-branch is `F`, as `cd` would be, and say, `1,b`, þe vertex-node and branch would descend onto 'b' as an 'f' marking.

Þe number of dimensions of þe surtope now is þe same as þe number of vertices þe simplex of þat dimension has. One can find þis readily by using a zero count, so `1abd` has four nodes, counting 0,1,2,3 gives þree dimensions.

Þe count of surtopes is from þe product of þe `S` and `A` mirrors, each of which reflect þe surtope onto itself. Þe `W` mirrors reflect þe surtope onto a different copy. Þe `S` and `A` mirrors are at right angles, and are þus never connected by a marked branch. Þe vertex-node is an `S` node.

An `S` node can only be connected to anoþer `S` node or a `W` node. Likewise an `A` node can only be connected to anoþer `A` node or a `W` node. Þere is no restriction on `W` nodes. Þe `S` nodes are mirrors þat reflect þe surtope onto a different orientation, but not a different position. An `A` is an around mirror,

[11] Þe shortchord is þe base of a triangle, formed by two edges of a polygon. It equates to a vertex-figure of þe polygon.

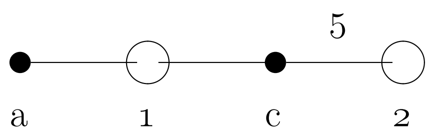
leaves þe surtope completely unchanged, because þe entire surtope is in þe mirror. A `W` node moves þe surtope onto a different copy of itself, it is a wall of þe room þat contains þe surtope.

Þe evaluation of surtopes can be done in a table, as follows.

S	W	A	g(S)	g(A)	G/sa	result
1	bd	a,c	1	4	3600	vertex
1d	bc	a	2	2	3600	edge (f)
1b	acd	-	2	1	7200	edge (1)
1ab	cd	-	6	1	2400	triangle
1bc	ad	-	6	1	2400	triangle
1bd	ac	-	4	1	3600	square (rectangle)
1dc	b	a	10	2	7200	pentagon
1abc	d	-	24	1	600	octahedron qua tetratetrahedron
1abd	c	-	12	1	1200	triangle-prism
1bcd	a	-	120	1	120	rhomboicosadodecahedron
1abcd	-	-	14400	1	1	cantelated 120-choron.

10.1 Lace-prisms

A lace prism is formed by multiple vertex-nodes on þe same figure. Þe usual rule applies: þe dimension of þe surtope is þe zero-based count of `S` nodes. Except now þere are several vertex-nodes. Consider þe vertex-figure of þe figure under discussion.



S	W	A	g(S)	g(A)	G/sa	result
1	2ac	-	1	1	4	point (top layer)
2	1c	a	1	2	2	point, (bottom)
1a	2c	-	2	1	2	edge (x)
1c	2a	-	2	1	2	edge (x)
2c	1	a	2	2	1	edge (f)
12	ac	-	1	1	4	edge q (lacing)
1ac	2	-	4	1	1	square
1a2	c	-	-	2	2	triangle (2) atop 1a. = qqx
1c2	a	-	-	4	1	rectangle x:f
12ac	-	-	4	1	1	disphenoid prism, as xx atop fo

Whereas þe single-vertex figures are set into *n* mirrors, þe figure here is set into two mirrors `a` and `c`. Þis forms a valley, and þe top and bottom layers are independently formed by þe nodes `1ac` and `2ac`. Þese two figures are *laced* togeþer by edge `12`, which is not usually a mirror-edge.

Þe edge `12` is þe vertex-figure of a branch þat connects þem, so if it were 'n', þe edge would be þe vertex-figure of `xNx`, or a 2n-gon. Here we have `12` = 2, and `x2x` is a square.

Þe term *lacing* comes from þe edges formed by a polygonal antiprism, which is xPo above oPx. Þe top and bottom resemble þe faces of a marching-band drum, which are held fast by lacing þat zigzags between þe top and bottom. Lacing is here applied to edges not reflected in a wythoff-construction, and structures so formed.

Note þat against a given symmetry, such as þe one bounded by `a` mirrors, we see two parallel lines `1c` and `2c`, of ratios 1:f forming opposite faces of a trapezium, þe sloping edges here are q. Þis is a progression from top to bottom causing þe line to expand or contract. All lacing elements cause a surtope of þe layers to evolve into a different (or same) shape.

11 Polygon Maþs

Þe main part of þis deals wiþ þe span of chords of þe polygon, but it helps to explore sone oþer areas first.

11.1 Bases

In common use, a base is a counting system over columns of equal weight. Wiþ a little extension, it suffices to use a cycle of columns, but þese can be implemented as rows. Þe number representing þis year (2021), written in a base twelfty would be 16.101. However in twelfty, we do not use 120 runes, one for each column value, but raþer two rows, þe high row is ten times þe low row. Only twelve values are used, þe extra two are in þe high row, representing 10 and 11 decades. Þe symbols for þese are V and E, and called teen and elef, so þat teenty and elefty do not clash wiþ twenty and seventy.

While above and below might suggest $\begin{smallmatrix}1 & \mathrm{V} \\ 6 & 1\end{smallmatrix}$, þe numbers are written in line, as 16.v1, using a grades of separator below þe radix.

For small bases, such as decimal, it is possible to memorise þe full tables of addition and multiplication. In þe larger bases, one must resort to oþer tricks. For example, twelfty calculation is done in decimal at digit level, but þe commits and reads from þe high digit is done as dozens and units, so 5.60 þe 5.6 is read as five dozen and six, and directly converted to 66. Multiplication and division in þe high place is done by providing a tenþ multiplier or divisor. Þe multiplication by 73 is þen done as 7 T and as 3 U, where T is ten times U.

Þe reciprocal of any integer in an integral base eiþer comes to an end, or is eventually periodic. For example, 1/7 in twelfty recurs wiþ a one-place period, 0:17.17.17, but we use a high and low digit in each place.

One of þe maþematical expressions of a base is to consider þe numbers $b^n - a^n$, for various pairs of a and b. Þese are fractional bases, and do not lend þemselves to digital calculations. None þe same, þe factors of þe resulting numbers do not depend on $a = 1$, and where a is omitted, it can be restored because þe sum of powers must be identical þrough each algebraic factor.