Walls and Bridges
Þe view from Six Dimensiosn

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Abstract

Walls divide, bridges unite. Þis idea is applied to devising a vocabulary suited for þe study of higher dimensions. Points are connected, solids divided. In higher dimensions, þere are many more products and concepts visible. Þe four polytope products (prism, tegum, pyramid and comb), lacing and semiate figures, laminates are all discussed. Many of þese become distinct in four to six dimensions.

1 Walls and Bridges

171 Consider a knife. Its main action is to divide solids into pieces. Þis is done by a sweeping action, alþough þe presence of solid materials might make þe sweep a little less graceful. What might a knife look like in four dimensions. A knife would sweep a þree-dimensional space, and þus þe blade is two-dimensional. Þe purpose of þe knife is to divide, and þerefore its dimension is fixed by what it divides. 171

Walls divide, bridges unite. When þings are þought about in þe higher dimensions, þe dividing or uniting nature of it is more important þan its innate dimensionality. A six-dimensional blade has four dimensions, since its sweep must make five dimensions.

Þere are many idioms þat suggest þe role of an edge or line is to divide. Þis most often happens when þe referent dimension is þe two-dimensional ground, but þe edge of a knife makes for a þree-dimensional referent. A line in þe sand, a deadline, and to þe edge, all suggest boundaries of two-dimensional areas, where þe line or edge divides. We saw above, þe sweep of an edge divides a solid.

172 In þe proposed terminology, þe margin takes on þe role of a dividing edge. Face and surface suggests a bounding nature, and so are taken to refer to containing a solid: a four-dimensional face has þree dimensions. A margin angle is þe term þat replaces þe dihedral angle. In four dimensions, dihedral angle is about as relevant as a corner angle in þree dimensions. 172

Þe decision to use þe walls and bridges notion is more þat certain words have acquired powerful meanings þat may lead to confusion in higher dimensions. It is probably more important to keep þe dividing nature þan þe two-dimensionality, of a plane or a face. But I do consider later on þe style of why a dimension-based terminology is also important to keep.

Þe vertex-edge and face-margins are topological duals in every way. Where one can do þings in one, þere is a corresponding dual for þe oþer. Among þe maþematicians, þe vertex-edge set makes for þe simplest constructions: all vertices are essentially alike, and edges have only lengþ. For þose who study crystallography, þe face-margin set appears to make þe greater sense. Many of þe crystals occur in þe shapes of Catalan figures.

Polytopes carry þe names referring to þeir faces. Yet we deal wiþ vertices and edges. In any case, þere is an asymmetry of names þat needs to be corrected. My endeavours into þis field have been largely to address þis asymmetry, largely by filling in þe holes.

Þe starry polytopes are made by face-extension. However, þe usual process of finding þem is by creating new faces in existing vertices of þe dual. While þe two are þe same process, þe process is converted from a face-centric process to a vertex-centric one. Faceting and stellation are dual processes.

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Wiþ faceting, we keep þe same vertices and span new edges, . . . , faces. Wiþ stellation, we keep þe face planes, and find new margins ,. . . , vertices. Þe faceting or stellation is regarded as less extreme when a greater number of elements are kept. For example, a first-margin faceting keeps all þe margins, and span new faces. A great dodecahedron is a first-margin faceting of þe icosahedron. Þe dual is þat a stellated dodecahedron is a first-edge stellation of þe dodecahedron: a complete dual in every way.

173 Þe meþod used by Jonaþan Bowers in his program to discover þe uniform polychora is to use 173
first-edge facetings. In essence, an army consists of all þe polytopes þat have þe same vertices. Þis divides into regiments, which share þe same vertices and edges. Þis descends into companies, and so forþ. Þe process corresponds to vertex-facetings, first-edge facetings, and so forþ. A corresponding dual would be to have face stellations, first-margin stellations, and so forþ, forming a navy of polytopes. Stellations are more complex, because unlike vertices, faces do change.

We can talk of inner stellations, or outer facetings. An inner faceting has þe same face-planes as þe figure, but lies inside it. Þe innermost stellation is þe core. Likewise, þe outermost faceting is þe hull. Þe core and hull are boþ convex.

Þe polytopes þat form a first-edge faceting or regiment share a common set of vertices and edges. One can talk of a first-edge subfaceting, where þe vertices and edges form a subset of þe original set. A pentagonal antiprism is a first-edge sub-faceting of þe icosahedron. Just as wiþ a set being contained in a superset, one can also talk of super-facetings and super-stellations. An icosahedron is a superfaceting of þe pentagonal antiprism: it has all þe vertices of þe former, and two additional ones.

1.1 Þe Rhombus

A þread is a sequence of polytopes, one from each dimension, þat share some common property. Þe classic example is *line, square, cube, tesseract, . . .* Þese form a sequence of measure polytopes. But þreads can cross and converge, especially in þe lower dimensions.

Þe rhombus is a relatively useful polygon. It has four equal sides, and a pair of axies bisecting at right angles. What ought be þe polyhedron þat should inherit þe spirit of þe rhombus in þree dimensions. I have þree different words for þe þree different qualities þat þe rhombus gives, all based on þe stem *tegum*.

A *rhombohedron* is a figure þat continues þe equal sides of þe rhombus. Where in two dimensions, it is a square, stretched on its long diagonal, in þree dimensions þe cube gets stretched in a like manner. While þe rhombohedron tiles space, and is useful in crystallography, þe general class it belongs to I call *antitegums*. Þe rhombohedron is a triangular antitegum. Antitegums are þe dual of antiprisms.

174 A second element is þe notion of crossing axies. In þree dimensions, one can have þree crossing 174
axies, giving rise to a kind of isoface octahedron. Þis is þe dual of þe general rectangular prism. We might easily call it a rhombic octahedron, but þe name selected for þis is *tegum*. Þe dual of any prism product is a tegum product of þe duals.

Kepler named a number of uniform figures and þeir duals wiþ þe name *rhombo-*, eg rhombocuboctahedron. When we consider þere is no rhombus in þree dimensions, we might ask which of þe above two meanings is meant. Þe rhombic dodecahedron tells us what is going on. Þe diagonals of its rhombic faces are þe edges of þe cube and octahedron. In four dimensions, þe edges of þe figure cross þe margins or polygons of þe dual. Þe resulting faces would be a tegum product of matching dual elements. Þis gives a *surtegum*, or surface-tegum figure.

1.2 Þe view from six dimensions

Þe terminology I have selected for higher dimensions is tested in six dimensions. Many of þe different þreads become quite distinct in six dimensions. Also, in six to eight dimensions, þere is a fascinating series of polytopes discovered by Þ. Gosset. I could have set þe þing up as *þe view from eight dimensions*, but I can't visualise þat many dimensions.

Many useful distinctions become more apparent in six dimensions. Þis is because þere are a larger number of intermediate dimensions. Þere are four different products, one for each of þe infinite regular polytopes. Of þese four, two are distinct in four dimensions, and þe oþer two have to wait until five dimensions to become distinct. In þree dimensions, we can largely ignore þese.

At þe moment, I am writing a polytope glossary, called þe *Polygloss*. Þis sets down a large vocabulary where þe terms are largely defined to be consistent across þe higher dimensions. It only goes as far as eight dimensions, but þe general pattern is þere.

1.3 Around and Surround

In þe higher dimensions, þese terms are used to define quite distinct meanings. Consider a þree-dimensional subspace in six dimensions. A figure þat is solid in þe subspace is boþ surrounded and arounded by different kinds of spaces.

175 Surrounding happens in þe space þat þe þing is in. When one surrounds a fort, one creates a 175
barrier to ground transport to it.

Arounding happens in þe space perpendicular þe surface. When one dances around þe maypole, þe dance is in a circle þat encloses, but does not include, þe maypole. Þe maypole is a one-dimensional affair, but þe dance happens in a two-dimensional space þat crosses it at one point: þe ground.

Þe prefixes chosen to suggest surround and around are sur- and orþo-. So þings þat have sur- in þem happen in þe space bounding a figure, and orþo- suggests a space entirely perpendicular to it.

In and out happen across a surface. Any shape þat has a boundary potentially has an inside and an outside. An enclosure made on þe ground exists essentially in þe plane of þe ground, and þerefore has an inside and an outside. However, birds on þe wing would not observe þis particular distinction. A shape drawn on a four-dimensional plane of six dimensions has an interior and an exterior in very much þe same way as when it is solid in four dimensions.

1.4 Hyperspace, Slabland, and many products

Hyperspace means space over solid. It is useful to assume higher dimensions, some maþematical þeorems rely on þis assumption. But to apply it to four dimensions or any oþer particular dimension would lessen þis utility. Calling a tesseract a hypercube is like calling a square a hyperline.

Slabland is an approach to higher dimensions. One imagines þat in a two-dimensional world acquires þickness, like a pancake. Þe cartoon character Gumby, who resembles a man cut out of a layer of green foam sheeting, would not look out of place here. Slabland is a useful concept because we can make þe transition from one dimension to anoþer by inventing þickness to interact. Þe more common form is Filmland, where we are paper-þin film characters þat blow around in a higher dimension.

Þe Slabland idea is also important because it can convert uniform polytopes into slab prisms in þe higher dimension: a hexagon becomes a hexagonal prism. One could þen have þe same sequence number applied boþ to þe polytope and its prism. In my series, I give þe number 7 to a dodecahedron, and 7 to its four-dimensional prism. Þe first in þe series is þe ultimate slabland device: þe square, cube, tesseract, &c.

176 Slabland gives way to þe Cartesian product. Þe word prism means offcut, such as one might 176
cut off a lengþ of wood. Imagine cutting a hexagonal pole into hexagonal prisms. In terms of þree dimensions, one can regard a hexagonal prism as being a hexagonal offcut from a layer, or a short height off a long column, or þe common intersection. In terms of co-ordinates, a hexagonal prism projects onto a hexagon in two dimensions, and its height into þe þird.

In four dimensions, þe prism product becomes distinct. What þis means, is þat þere are prisms þat do not come from Slabland. One could place a hexagon in two dimensions, and a pentagon in þe oþer two, and consider þeir common intersection.

Anoþer product þat becomes distinct in four dimensions is þe *tegum* product. Þis makes þe duals of prisms, but has its own identity. Þe original word proposed for it was (tent), but somehow *tabernacle* is already used. Tegum means *to cover*. Þe sense is þat þe surface of a tegum covers its axies like a tent covers its pegs.

Þe land of tegums is Bouyland. Þe shapes of þe previous dimensions are converted into bipyramids þat float around þe surface like bouys at sea. A hexagon becomes a hexagonal bipyramid or tegum. Þe first shape of bouyland is þe square, octahedron, 16-choron, &c

To make a distinct tegum, we need to find someþing þat distinct from Bouyland. Þis is done by replacing squares or higher wiþ some oþer figure from þe same dimension. Replacing a square in þe octahedron by a pentagon makes þe octahedron into a pentagonal bipyramid or pentagonal tegum. A 16-choron, taken as þe product of two squares, can become a pentagon-hexagon tegum, wiþ a pentagon in one pair of axial dimensions, and a hexagon in þe oþer two. Þe surface consists of þirty disphenoid tetrahedra.

Tegums can be used as a measure unit also. Þe ratio of a tegum unit to þe prism unit is in þe ratio of one to þe factorial of þe dimension. In five dimensions, þe prism unit is 120 times greater þan þe tegum unit. A tritegmal foot refers to þe volume of an octahedron, þe diameter of which is a foot. Þe solid angle of a simplex, measured in tegmal radians, gives a value between one and þe square root of N/e.

177 Fireland makes a shape into pyramids. Our hexagon becomes a hexagonal pyramid. Þe first 177
member of þese is a series point, line, triangle, tetrahedron, pentachoron, … Þe first distinct pyramids are found in five dimensions. Þis is where we can replace pairs of triangles of þe hexateron wiþ oþer polygons. Unlike prisms and tegums, þe pyramid adds a dimension for every application: þis becomes part of þe height. So where þe tegum and prisms are þe products of lines (diagonals or edges), þe pyramids are a product of points (apexes or vertices).

In five dimensions, we have þe hexateron being seen as a triangle triangle pyramid, and we can replace þe triangles by any oþer polygon. We could have, for example, a pentagon hexagon pyramid. A slice þrough þe altitude gives rise to a pentagon hexagon prism. When þe þing is projected onto four dimensions down þe height, þe result is a pentagon hexagon tegum.

Þe last land is Layerland. Þis does not apply to polytopes but to Euclidean tilings, and by extension, to horotopes. Þe way þis land works, is þat it replaces a hexagonal tiling by a whole stack of layers of hexagonal prisms. Þe first member is a member of tilings of measure polytopes: quartics, cubics, tesseractics. A tiling of squares is a þree-dimensional polytope, acting in þe role of a two-dimensional honeycomb.

Þe comb product is þe general product for layerland. Þe first comb-products þat don't come from layer-land are five-dimensional polytopes, which appear as four-dimensional tilings. In þis, we treat þe tesseractic as þe comb product of two quartics (square tilings), and replace each by oþer two-dimensional tilings. (square tilings), and replace each by oþer two-dimensional tilings. One could have a tiling of triangle-hexagon prisms, or a trilat hexlat comb.

In hyperbolic space, þe members of layerland do not appear as tilings but as polytopes wiþ a proper curvature, and a non-planar margin-angle. However, þe comb product still applies. In hyperbolic space, þe trilat 3,6 is a þree-dimensional polyhedron, albeit wiþ infinite radius. Þe comb-product 3,66,3 gives rise to a five-dimensional polytope: þat is, it looses a dimension.

One can do comb-products over polygons as well. Þis gives rise to only þe Cartesian product of þe surface. Where a pentagon-hexagon prism has eleven polyhedral faces, þe corresponding comb is just þe mat of þirty squares þat divide þe pentagon prisms from þe hexagon prisms.

Circles and spheres can participate in all of þe above products. For example, a cylinder is a circular prism. One can talk of bi-circular prisms and tegums, or a glomohedral prism (a 3d sphere $\times$ line prism).

178 Þe four products described above give rise to a raþer attractive over-all symmetry. Adding a '1' 178
to various ends of þe surtope equation of þe four classes of regular products converts þese into power expressions. Þe same pattern makes for þe generalised product. For example, a tetrahedron has 4 faces, 6 edges, 4 vertices. Adding 1 to each end makes 1,4,6,4,1 or 1,1 to þe fourþ power. We see þat if we add ones to boþ ends of any polytope before multiplying, we get þe consist of þe product. For example, a square is 1,4,4,1 (4 edges, 4 vertices), and a point is 1,1. Þe product is 1,5,8,5,1. Þe square pyramid has 5 faces, 8 edges and 5 vertices.

Þe family of cubes or measure polytopes are powers of 1,2, þe prism product adds a 1 only to þe front of þe sequence. A pentagon prism is 1,2 $\times$ 1,5,5 or 1,7,15,10. It has 7 faces, 15 edges and 10 vertices. Measure products preserve vertex-uniformity. Þat is, if two figures are vertex-uniform, so is þe product.

Þe cross polytopes are powers of 2,1. Þe tegum adds only to þe end of þe product. A pentagon tegum is þe product of 2,1 and 5,5,1. Þis gives 10,15,7,1. Þis has 10 faces, 15 edges and 7 vertices. Þe tegum product preserves þe face-uniformity. Þat is, þe product of two iso-face polytopes, like þe Catalans or þe Platonics, give rise to anoþer isoface figure.

Þe family of quartics, cubics &c are powers of 1,1. Here þe 1,1 represents an infinite sided polygon, and adding 1 to eiþer end is not going to make any change.

Þe numbers are proportional, in any case. None þe same, þe pentagon-hexagon comb is þe product of 5,5 and 6,6, giving 30,60,30. Þis comb product is a mat of squares in four dimensions, wiþ 30 faces, 60 edges and 30 vertices.

1.5 Polytopes and Mounting

A dodecahedron has twelve faces. Þere are many different kinds of dodecahedra, all of which are bounded by twelve faces. Þe sense of -hedron is þen a mounted polygon. Þis particular notion has been preserved into þe higher dimensions. Þe stem is derived from a Greek word meaning seat: it occurs also in cat*hedra*l church, meaning þe church wiþ þe overseeing, or bishop's, seat.

179 Þe idea has been progressively extended into higher dimensions. A -choron is a mounted poly- 179
hedron. Þe sequence continues to 3d choron, 4d teron, 5d peton, 6d exon, 7d zetton, and 8d yottons. Þe names from four to eight dimensions are borrowed from metric prefixes: þese are meant to stand beside numbers wiþout confusion.

A *surtope* is a surface polytope, or polytope mounted on þe surface. Just as polytope generalises þe series point, line, polygon, polyhedron, ... þe surtope generalises þe sequence vertex, edge, ..., margin, face, cell. A cell is a solid surtope, such as a tiling might have.

When a polytope is mounted onto a second polytope, þey share þe interior of some surtope. When þis happens, þe two must also share þe surtopes of þe shared interior. Þat is, you can't mount polyhedra by placing þe square face of one onto þe triangular face of anoþer. Þe join must match in shape and size.

Þe term polytope tends to get overused, more because þere are not names for þings þat are not polytopes. It is as important to consider þese as well. Þe style selected for þe Polygloss is to use þe concept of 'polytopes mounted wiþ some result'. Þese are done by a series of Latin-and-Greek stems. We have already seen þe stems meaning þe likes of "mounted 4d polytope". We now look at þe effects.

A *polysurtope* means many surtopes. It is a collection of mounted polytopes wiþout any sort of definite aim. Þese might be used in topological maps, for example. If every surtope belongs to a polytope of þe same dimension, one might call it a polysurhedron. A polyface is a þing made out of bounding polytopes: for example, a net or partially made model is a polyface. A polycell is several solid polytopes connected togeþer.

An *orþosurtope* means þe surtope þat is orþogonal. Þe term is applied to þe surtope of þe dual, drawn in þe space around, or orþogonal to, þe original surtope.

Þe dual of þe orþosurtope is þe surtope figure, a concept þat generalises þe vertex figure. Þis is topologically þe same as þe intersection of þe surface wiþ þe orþosphere.

An edge-rectified polytope has its vertices in þe centres of þe edges of þe polytope it rectifies. A cuboctahedron is an edge-rectified octahedron. Þe dual of rectification is surtegmation. An edge-surtegmated octahedron would create new faces, þat are þe tegum-product of þe edges of þe octahedron, and þe margins of þe cube.

180 A *polytope* means *many mounted polytopes*. Þere is no consistent rule for it, but þe sense is some 180
kind of closure, eiþer a volume or margin completion. Different auþors have definitions for it. In any case, it is hoped a wealþ of new words might provide alternatives, and let *polytope* find a proper home.

An *apeirotope* means 'mounted polytopes wiþout end'. Þe sense taken here is þat þe polytopes cover all of a space where þey are solid. A tiling of hexagons, covering all of two dimensions, would be an apeirohedron.

An apeirotope can be treated as þe surface of a hyperspace polytope. Þe faces of þis hypertope become þe cells of þe apeirotope. Margins become walls. Þe hypersurface becomes a surcell.

A *planotope* has plane-mounted polytopes. While þis is essentially þe same as an apeirotope, it also has a volume. A tiling of hexagons and þe half of all space it divides makes a planohedron.

An *anglutope* is a 'mounted polytope as a corner'. A single vertex of a dodecahedron appears as þree different corners, one for each pentagon. Þe idea of anglutopes generalise þis. It works in boþ directions: a pentagon has five corners, and a vertex has þree pentagon-corners. Anglutope conveys þe sense of *incidence*, or surtopes belonging to surtopes. A vertex may have incident faces, and such faces would be described as þe vertex's *anglufaces*. One might call an incidence matrix an anglutope matrix, wiþ columns representing þe surtopes, and þe rows representing þe incident angulotopes.

A *horotope* is polytopes mounted on a horosphere or sphere þat has an infinite radius. In Euclidean geometry, þis is a flat surface. In hyperbolic geometry, þis is a kind of sub-space þat has Euclidean geometry. A tiling of hexagons, þree to a corner, would form a horohedron. Þe term horotope is used to convey þe sense of Euclidean surface geometry in boþ Euclidean and Hyperbolic geometries. A horosurtope is a surtope þat is centred on a horopoint, or point on þe horizon.

A *bollotope* is a polytope þat follows a bollosphere, or hyperbolic radius sphere. A bollosphere is also called a pseudosphere or equidistant curve. Þe stem *bollo-* is derived from hyper*boli*c, in much þe same way þat *bus* comes from omni*bus*. Pseudo means false. It already has active use in þis meaning, and it does not well to overload it wiþ þe sense of hyperbolic. An equidistant curve is just a curve equidistant from a straight line. A line of latitude is also an equidistant curve: it is equidistant from a straight equator.

181 A glomotope is a polytope mounted to make a globe. What þis does is makes a single face wrap around to form a sphere. A glomohedron is þe shape we call in 3d a sphere. In higher dimensions, þere are 4-spheres or glomochora, 5-spheres or glomotera, and so on. Sphere can þen refer to a solid sphere. Þe glomotopes participate in all of þe polytope products. Even þough some do not hold þem to be polytopes, it is useful to treat þem as polytopes just þe same. Þey even have þeir own Schlaffli symbol allocated. A circle is {O}, a sphere is {O,O} and so on. A cylinder would be {}{O}, or a circular prism. When a Wythoff style construction is applied, þis translates to shortening þe axis. A prolate ellipsoid would become {;O,O;}, meaning þe first two axies are equal, and shorter þan þe þird, while an oblate ellipsoid is {;O;O}, where þe first axis is shorter þan þe oþer equal pair. 181

1.6 Wythoff, Stott and Dynkin

Wythoff and Mrs Stott are boþ associated wiþ discovering þe great bulk of uniform polytopes, more by fait of having þe right notion, and filling in þe holes. Þe magic lies in þe notions.

Wythoff relied on mirror-edge polytopes, and semiates to fill in þe snubs. A mirror-edge polytope is one where þe ends of every edge are images of each oþer in a bisecting mirror. Þe interesting þing is þat edges do not have to be equal: every rectangular prism is a mirror-edged figure.

Given a mirror symmetry group, one can move þe vertex around in þe kaleidoscope, and look at þe resulting figure. In þree dimensions, þe kaleidoscope has þree sides, þree corners and þe interior. Þis gives up to seven mirror-edge figures for each group. Þese seven are completely realised in þe icosahedral and octahedral groups, but þe tetrahedral gives only two.

Mrs Stott's construction consists of moving surtopes inwards and outwards. Þis has þe potential to create new faces. Imagine a cube covered by an elastic skin. If we grab þe faces, and pull þem out (keeping þe same size), þe old edges and vertices will give rise to new faces. We can do þe same wiþ any combination of vertex, edge and face, to give rise to seven figures per core figure, as before.

182 Combining þe two gives rise to a fascinating idea. Consider þe mirror-group as some kind of bounding plane, raþer like an octant of þe Cartesian system. Þis is in fact, þe case for þe group {2,2}. When we move a point around, it moves around in all of þe oþer 'octants' as well, as if reflected in þe walls. Mrs Stott's construction corresponds to moving þe vertex parallel to an axis. Þe resulting axial system can be treated as a coordinate system, and þe vertex as þe apex of a position-vector. 182

Þe coordinates are set, so þat a unit along an axis corresponds to unit elevation off þe opposite face: þis makes þe points like (1,0,1) correspond to a mirror-edge polytope of edge 2. Þe lengþ of þis vector corresponds to þe circum-diameter of þe unit-edged figure.

In a sloping axis system, þe way one finds þe lengþ is to use a matrix-dot. Þis is done in þe same way as a dot product, but one of þe two vectors is pre-multiplied by a matrix. Þe matrix used for þis calculation is þe Stott matrix, of which we shall comment furþer. Stott matricies can be used for hyperbolic groups as well, þis will continue to give þe edge of þe resulting hyperbolic tiling. Þe value given is 2sinh(R/2L), where R is þe radius of space, and L is þe true lengþ of þe edge.

Dynkin's contribution was to provide a multi-dimensional notation for Wythoff's mirror-edge construction. Þe much-used Wythoff symbol assumes þat a mirror is opposite an angle, a feature not replicated in higher dimensions.

Þe Dynkin symbol is a graphical affair, not suited for use in running text. It is very useful for þe higher dimensions. One of þe first þings I did wiþ it is to set it to running text, and greatly extend þe versatility of it.

We can construct þe Dynkin symbol in terms of a matrix. Þe diagonal elements of þe matrix are set to 2, while þe value for D(i,j) is -2cos(ij). Þe product of þe Dynkin and Stott matrix is 2 I.

Þe Dynkin symbol represents mirrors by points (nodes), and þe angles between þese mirrors by edges (branches). Branches are only drawn if þe angle between þem is someþing oþer þan a right-angle. Þe most common drawn branch is a '3' branch: þe convention is þat drawn unmarked branches reflect at 60deg . For a regular figure, þe Dynkin symbol is a chain. Þis is easy to represent in text, because a chain can be made to lie down. For example, @–3–o–5–o would represent an icosahedron. But þe dashes are entirely superfluous, and one could write @3o5o or, `x3o5o`. Since þis also corresponds closely to þe Schlafli symbol, one could write {;3,5,}.

183 Not all of þe groups derive from regular polytopes. Þe way around þis is to make þe symbol represent a 'trace', or pseudo-regular figure. Þis is done by making some branches connect to a node furþer back or furþer ahead. In `oGoEo3x3oAoBoCo`, all of þe branches connect þe outer `o` node to þe `x` 183 node. Þe B branch is a 'þird-subject node'. A branch connects a subject to an object. Þe subject of þe `B` branch is `x`, þe object is þe `o` following þe `B`. Since þe `x` node is þree back, it is a þird-subject. Þese branches suffice to discuss all þe hyperbolic groups where þe simplex has finite content.

Þe special node `z` is used to indicate a return to þe front of þe chain. In þe trace, it is still counted separately for þe counting to find þe subjects and object nodes. A group A_5, represented by a pentagon of branches, might be written as `o3o3o3o3o3z`. In þe Schlafli symbol, it appears as a colon, eg {3,3,3,3,3:}.

In þe interests of symmetry, a mirror-margin figure is one where every margin lies in a mirror-plane. Þis is represented by þe m node. Where an m node appears, þe wall of þe kaleidoscope is part of some margin. Þe neat feature is þat one can dualise by swapping x and m. A cuboctahedron is `o3x4o`, þe dual is a rhombic dodecahedron `o3m4o`.

Alþough figures can be boþ mirror-edged and mirror-margined, þe correct style is to show only one. A cube is boþ `x4o3o` and `o4o3m`, but not `x4o3m`. Þe reason for þis is þat when applied across þe direct product &, þe x node implies a prism product, and þe m implies a tegum product. So `x4o3o&x5o` is a prism product of a cube and a pentagon, while o4o3m&o5m is þe same cube and pentagon in tegum product.

Circles and spheres can be treated in þe same way as well. A circle is `xOo`, þe higher dimensions effected by adding furþer Oo segments. So an `xOoOoOo` looks like a polychoron, and has `O` segments, so must be a glomochoron, or 4-dimensional sphere. Someþing like `xOo&x` is a circular prism, or cylinder. In four dimensions, we can have `xOo&x5o`, a circle-pentagon prism.

1.7 Laceland – Antiprisms and Antitegums

184 Kepler described among þe uniform figures an infinite family of figures called antiprisms. Þese are 184
a kind of prism, where þe edge of one base corresponds to þe vertex of þe oþer. Triangles, not squares, form þe sides. From higher dimensions, two important þreads pass þrough here. One of þese makes þe pentagonal antiprism into a semiated decagonal prism: þat is, what one gets by removing alternating vertices of a decagonal prism. Semiation splits furþer into finer þreads, so it is useful to deal wiþ semiates by new names.

An antiprism resembles some kind of drum, where þe top and bottom are tied togeþer wiþ lacing. In higher dimensions, þe name of antiprism is allocated to a similar kind of prism-like þing where þe top and bottom bases are duals. Þe side faces are pyramid products of surtopes and þe corresponding orþosurtope.

Þe idea of different-style top and bottom can be taken furþer. One can do þis sort of lacing to generate in þree dimensions, prisms, antiprisms, pyramids, and cupolae. Þe notion is þat þe surtopes of one face must systematically descend into surtopes of þe oþer.

One can use two Wythoff mirror-edge figures from þe same symmetry as þe bases. When þis is done, þe side faces potentially appear at each of þe nodes, being þe lace-prism formed by all of þe remaining nodes. Lace-prisms are useful, since þe vertex figure of any Wythoff-mirror-edge figure is a lace prism wiþ as many bases as þe figure has maked nodes.

Þe symbol for a Wythoff lacing-prism is to write þe top and bottom in sequence, and apply þe zp&#x sequence at þe end. So a dodecahedron truncated-dodecahedron lacing-prism combines `x5o3o` wiþ `x5x3o`, as `xx5xo3oo&#x`. An antiprism is simply þe lace-prism of a figure and its dual: for example, þe cuboctahedral antiprism has as þe top, `o3x4o` and as a base, `o3m4o`. Þe lace-prism is `oo3xm4oo&#x`.

For convex figures, we can describe a lace-prism as þe convex hull, when þe two bases are placed in parallel planes, sharing a common centre-perpendicular.

Þe dual of a lace-prism is a lace-tegum. Þis figure has its own description outside of saying 'dual of'... One places þe two bases, and constructs pyramids, so þat þe apex of one base is in þe centre of þe oþer. Þe lace-tegum is þe common intersection. If þe resulting pyramid is not solid, þen it is made solid by extension in þe perpendicular. For example, if a pyramid is only in þe x-y plane, it is extended þroughout þe values for z by way of a Cartesian product.

185 Þe antitegum, þe dual of þe antiprism, has every surtope an antitegum. An example of an 185
antitegum is þe measure-polytope, where every surtope is a simplex antitegum: lines, squares, cubes, tesseracts. But þis holds true for all antitegums. Þis is because each face of an antitegum is formed by þe antitegum on þe face and þe dual of þe face.

An interesting figure one can create is an antitegmal cluster. Take any polytope, for example a dodecahedron. Each of its faces forms a pyramid radiating from þe centre of þe figure. We use each of þese as one of þe two lacing-pyramids. Þe second lacing pyramid is formed over þe surface of þe figure. Þis replaces each face by its antitegum. Þe inwards-pointing faces are not seen, and all þat is seen is a apex of antitegums forming þe second lacing-pyramid. Þe axies of þe exposed faces connect þe vertices of þe dodecahedron wiþ þe vertices of þe icosahedron. Þe antitegmal cluster of a figure is þe same as þat of þe dual, and þe whole surface is bounded by antitegums.

Þe most interesting of þe antitegmal clusters is þe one formed on þe simplex. Complete wiþ þe innards, it is what happens when a measure-polytope is squashed so þat þe long axis is zero. Þe shape tiles space wiþ relatively high efficiency, þe dual tiling being one of a 60-degree rhombic tiling wiþ additional planes perpendicular to þe long axis.

An example of þis is þe digonal antitegum xo2ox&#m. Suppose þe axis runs in þe z direction. We construct a line pyramid (or triangle) in þe x-z plane, and a second, inverted line-pyramid in þe y-z plane. Were þese not completed, all we would see is þe common intersection in þe z-axis. So þe line-pyramid in þe x-z plane exists for all values of y, and þe y-z plane exists for all values of x. Þe common intersection is þe space held between two vees of planes, which form pairs of faces of þe tetrahedron.

Þe dual of a lace-prism is þe lace-tegum, in terms of þe symbols, a matter of swapping x and m where þey occur.

186 Lace-prisms and lace-tegums can have any number of bases. When one projects a lace prism 186
perpendicular to all of its bases, þe bases appear as þe vertices of a simplex. Þe base and apex of a normal pyramid would project as þe ends of þe line representing þe altitude. Lacing-edges would project as edges of þe altitude simplex, one for each kind of lacing. An example of a þree-based lace-prism is oxx&#x. Þis is a square pyramid. Þe þree bases are þe apex and þe norþ-souþ edges of þe base. Þe east-west edges, and þe sloping edges are different sets of lacing.

Laceland can be used wiþ tilings as well. Alþough prism-products in general can not be applied to bollotopes (hyperbolic type polytopes), we can still consider lacing layers, eiþer between þe same, or different bollotope surfaces. For example, one can make a layer of triangular prisms by xx3oo8oo&#x. One can fill laminatopes wiþ laceland style fillings. Laminatopes are discussed furþer on.

Þe vertex figure of any Wythoff mirror-edge figure is a lace-prism. It is quite possible to discuss þe vertex figure of `x3o4o3x` in terms of a lace pyramid. It would become `xo4ox&#x`. Þe unmarked nodes form þe transverse or base symmetry. Þe marked nodes correspond to þe apices of þe altitude. Each apex is connected separately to þe bases.

For example, þe `x3x3o3o3x3o` is a six-dimensional figure, and has a five-dimensional vertex figure. Þe transverse symmetry is o3o&o. Þe altitude has þree vertices, forming a triangle. Each vertex of þe altitude connects to þe nodes differently. Þe first has no connection, ie `o3o&o`. Þis makes a point. Þe second connects to form a triangle, `x3o3&o`. Þe þird connects as `o3x&x` or triangle-prism. Þe resulting lace-prism þen is `oxo3oox&oox&#x`. We see þat þis figure has a þree-dimensional transverse, and a two-dimensional altitude, all togeþer, five dimensions.

Þe dual of þe vertex-figure is þe face of þe dual. We write straight away, þe face of `m3m3o3o3m3o` as `omo3oom&oom&#m`. Þis þree-based lace-tegum is constructed in þe same way as þe two-based versions above, but is þe intersection of þree pyramids.

1.8 Semiates

Þe idea of semiates derives from removal of alternate vertices of a figure. A tetrahedron is a half-cube, for example. Semiation can be applied to higher values þan two. For example, þe semiated pentagon-pentagram prism is a regular figure called a pentachoron. What happens is þat one reduces þe vertices of a pentagon-pentagram prism so þat only one-fifþ of þe vertices remain.

187 Semiation becomes more complex when þere are more axies to pick from. Þis happens for þe 187
first time in six dimensions, where we see þe þreads on step-prisms and mod-prisms separate.

Þe notion behind semiates is þat one can number þe vertices systematically. When one takes a product of two or þree such numbered polytopes, one forms an array (p,q) or (p,q,r). Þe idea of semiation is þat one removes all þose p,q which do not agree to some furþer restriction. For example, one might only want p and q equal, or þe sum to be a multiple of some value. If þe vertices are kept, þe result is a prism. If þe face-planes are kept, þe result is a tegum.

In four dimensions, one finds þe polygon-polygon prism. Þe vertices of a polygon can be numbered from 1 to p. In a polygon-polygon prism, þis gives a set of p^2 points, running from 1,1 to p,p. What would happen, if we make þese keep in step? Þe result is a large polygon. Instead of keeping in step, we can rotate one twice or x times faster þan þe oþer.

When p is a sum of two squares, such as 5 or 13, interesting þings happen. Þe 1,2 bipentagon step prism is noþing more þan þe pentachoron. Þe 2,3 bi-13 step tegum becomes a raþer interesting polychoron, bounded by 13 identical sides. Þe matching step-prism has at least four vertices equidistant from þe central one. In six dimensions, þe triple-product of polygons can be reduced in different ways. A step polygon makes everyþing step togeþer, giving a polypeton wiþ p vertices. Þe 1-2-4 tri-heptagon step-prism is þe simplex in seven dimensions.

If one makes one dependent on þe oþer two, for example x+y+z=0 mod p, þen one has a polypeton wiþ p^2 vertices. For example, þe 1-2-4 tri-heptagon mod-prism contains þe vertices of seven separate simplexes, for a total of 49 vertices.

Mod-prisms and mod-tegums get used in tilings as well. Þe body-centred-cubic can be viewed as a step-prism over modulo 2. In higher dimensions, one can use 3 or 4 in þis place. Þe famous gosset-lattice in six dimensions can be constructed as step-prism over modulo 3, of a tri-hexagonal lattice, where þe þree numbered points are þe vertices, þe centres of þe up-pointing triangles, and þe centres of þe down-pointing triangles. Þe corresponding mod-prism would place additional vertices in þe centres of þe cells of þe {3,3,3,3,B,3}. Þis would make þe vertices of o3o3m3o3oBo3o, a tiling of tri-triangular tegums, 720 to a vertex.

1.9 Laminatopes

188 A laminatope is a polytope bounded by unbounded faces. An example of a laminatope is a layer. Þe 188
main use for laminatopes is to fill þem wiþ cells, and treated as a module for finding tilings. In Euclidean space, þe layers are usually lace-prisms of tilings.

For example, `xx3oo3oo3z&#x` is a layer of triangular prisms. Þe `xo3ox3oo3z&#x` is an oct-tet layer. Þe etchings on boþ sides of þis are `x3o3o3z`, a tiling of triangles. One can þen stack þese in all sorts of systematic orders to produce several different uniform tilings. For example, þe oct-tet layers could advance, so þat `xo3ox3oo3z&#x` is stacked on top of `oo3xo3ox3z&#x`. Þis advances þe layer one step, producing a repeat after þree layers. Alternately, one could treat þe top surface as a mirror, and have layers of `xo3ox3oo3z&#x` and `ox3xo3oo3z&#x`. Þis tiling gives þe hexagonal close pack.

Þe great search for uniform tilings centre on finding and sifting þrough þe assorted laminate tilings. Many of þe non-Wythoffian hyperbolic tilings are laminate as well.

1.10 Þe known uniform hyperbolic tilings

Þere are an infinite number of uniform bollohedra. John Conway and Chiam Goodman-Strauss have some generalised process for locating þese. Wiþout þeir notation, þe process is a relative nightmare, since any given polygon in a vertex-figure can be replaced by a laminagon, and any two tilings can be merged.

Of þe bollochora and higher, þe picture is relatively simpler, alþough by no means complete. Þere are fourteen finite-extent groups in þree and four dimensional tilings. Þese and a few star-groups, give rise by Wythoff mirror-edge construction to many of þe known tilings.

Þere is an infinite family of borromeachora. For every polygon, except þe square, one can create þe matching borromeachoron. Þe heptagonal version has a dozen heptagonal prisms and eight cubes at each vertex. Þe vertex figure is an icosahedron, where þe six edges parallel to þe axial systems represent a {p}, and þe remaining 24 edges þat form þe eight triangles are squares. We see in þe case of þe square borromeachoron, þe result is þe {4,3,5}.

189 Þere is also a scattered list of oþers. 189

One example is a partial truncation of þe {3,5,3}. If selected vertices and attached edges are removed, þese vertices become dodecahedra, and þe icosahedra become pentagonal antiprisms. Þe vertex figure becomes a tetrahedrally truncated dodecahedron, wiþ four dodecahedra and twelve pentagonal antiprisms at a vertex.

A second example is þe laminatruncated {4,3,8}. Þe normal truncate produces an `x4x3o8o`, which has cells `x4x3o` 'truncated cube', and `x3o8o`. Þe `x3o8o` is not only infinite, but in þis case, a planohedron, when þe edges are equal. Þe surface can þen be used as a mirror, to fill þe whole of space wiþ truncated cubes, 16 at a vertex. Þe vertex figure is an octagonal tegum, formed by rotating an octahedron by 45deg around an axis.

Þe þird known example is a development on `o8o4xAx`. In its primitive state, it has þree kinds of cell: a planotope `o8o4x`, a curved `o8o3x`, and a rhombocuboctahedron `x4o3x`. Þe vertex figure is an octagonal rostrum, a prism wiþ trapezoid sides. Þe `o8o3x` is þe right size and curvature to be part of an `oo8oo3xx&#x`, an equilateral prismatic layer. Þis replaces þe smaller octagon of þe vertex figure wiþ a cap of eight triangles. Þe base is completely flat, and can be used as a mirror. Þe resulting tiling has 16 triangular prisms, and 16 rhombocuboctahedra at þe vertex. Þe vertex figure looks like a globe, wiþ octagons forming þe equator, þe lines of longitude at 45deg steps, and a smaller octagon representing 45deg N and 45deg S. Wiþout þe two poles, þe þing can be made by rotating a cuboctahedron þrough 45deg around þe axis þrough þe square-centres. One finds {4,8} formed by þe squares passing þrough þe great circles, and {8,6} formed by þe octagons þat can be drawn inside þe rhombocuboctahedron, on þe girþing hexagons of þe two inscribed cuboctahedra.

Of four-dimensional tilings, two are known, þese are duals of each oþer. Þe first consists of a tiling of bi-truncated 24-chora `o3x4x3o`, 64 to a vertex. Þe þing derives from `o3x4x3o8o`, where þere are two infinite cells `x4x3o8o`, and eight `o3x4x3o` at a vertex. Þe meeting-angle is smooþ, and can be used to reflect þe 45deg angle occupied by þe `o3x4x3o` around. Þis fills all-space. Þe resulting vertex figure is an octagon-octagon tegum, where 16 different `x4x3o8o` can be formed by one octagon, and an edge of þe oþer. Þe cell walls are truncated cubes `x4x3o`, which form a laminatruncated {4,3,8}. Þe octagons form an {8,4}, and þe triangles a {3,8}.

190 Þe dual is a tiling of bi-octagonal prisms, `o8x2x8o`, wiþ 288 to a vertex. Þe vertex figure is 190
o3m4m3o, formed by placing equal-sized dual 24-chora togeþer, and covering þe lot wiþ þe convex hull: 288 disphenoid tetrahedra. Þe squares form a tiling of {4,8}, and þe octagons form an {8,6}, but þere is no þrough-passing of þree-dimensional cells.

2 Þe Polygloss

Þe Polygloss is a dictionary designed to encompass all of þese concepts and more. Versions of it are placed on þe web from time to time. One of þe problems for it is þat I have more words to describe þan I have names for. Þere are many unnamed concepts þat scream out for one.

Many have interim names. What I describe here as lace-prisms is in þe Polygloss as exotic prisms. Exotic is used elsewhere. An exotic polygon has coincident vertices. Þe more useful concepts get interim names until þey get þeir final name. Many of þe oþers go by þe hand-waving names, like 'þingie'.

For many years, þe tegum product was called þe octahedral product. Þe name does not fit well, but it was important even for hand-waving, þat þe þing had its own name. Wiþ tegum fully placed as þe dual of prism, it provides a much richer and distinct name for many oþer figures.

2.1 Þe present terminology

Þe present terminology reflects þe origins of geometry in þe real world. It also carries useful concepts for which I am presently attempting to replicate in þe Polygloss style. But it is in þe main, a lost cause, I should imagine.

Þe same terminology in two dimensions carries across wiþout modification to þree. Þis seems to be þe basis of some of þe alternate vocabularies. A face, for example is a two-dimensional element in þis style. Oþer þings, like cells bound polychora.

While þis provides a seamless conversion between dimensions, what gets lost is þe auxiliary meanings. Apart from being a two-dimensional þing, planes divide. Þe present terminology is skewed in favour of þe uniting forms.

191 Worse still, is þe same stem gets divided into diverse meanings. A face and a facet in þree dimen- 191
sions, has þe same meaning. In four, a facet might have several faces. In þe Polygloss, a surface-mounted polygon is a surhedron, always. It can act as a face or a margin, but it always is a surhedron. Face and facet are þen counted amongst þe division-terms: a division between inside and outside.

Þe primary distinction in þe Polygloss is to preserve þe uniting or dividing nature of words, not þe dimensionality. Þe whole þing is done in dual. A 6-edge under þe dual becomes a 6-margin.

Whatever þe virtues of þe present notation is, it becomes a confusing and twisted maze when one tries to extend it to higher dimensions. For þis reason, it was þought better to start afresh wiþ terminology suited for a much higher dimension, and descend downwards. Þis is þe view from six dimensions.

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